Université de Paris - École Doctorale 386
Laboratoire de Probabilités, Statistique et Modélisation

Thèse de doctorat

Discipline : Mathématiques

Sparse random graphs: from local specifications to global phenomena

Guillaume Conchon--Kerjan

Sous la direction de Justin Salez

2017-2021

Remerciements Ce manuscrit résume un voyage de plusieurs années, qui m'aura permis d'explorer différentes questions des probabilités mais surtout mes goûts, mes capacités et mes limites. Voici quelques mots pour ceux qui y ont joué un rôle essentiel.
Je tiens en premier lieu à remercier Justin. Depuis le projet enthousiasmant que tu m'as proposé en janvier 2017, ton intuition et nos échanges m'ont ouvert de nombreux horizons mathématiques. Que ce soit pour me guider, me relire ou m'aider à m'organiser, tes conseils se sont révélés remarquablement efficaces. Je les garde précieusement pour la suite de ma carrière et pour pouvoir, j'espère, les transmettre à mon tour.
Je suis également redevable à Christina. J'ai beaucoup appris de ton expertise et de ton expérience depuis ce stage à Oxford en 2016, puis durant notre collaboration plus longue et plus riche que prévu.
Je remercie Grégory et Perla d'avoir rapporté ma thèse. C'est un honneur d'avoir pu bénéficier de votre travail de relecture, et de vos nombreux commentaires instructifs, pris en compte dans cette version révisée. Merci également à Bénédicte, Nicolas, Jiří et Nathanaël d'avoir accepté de faire partie du jury. Je suis heureux que mes travaux puissent avoir le privilège de votre attention.
Le LPSM m'a fourni un très bon cadre pour ma thèse. Merci Mathieu pour tes conseils et nos discussions. Merci Cyrille pour tout ce que j'ai appris sur l'enseignement après quatre ans en tant que chargé de TD de tes cours. Merci aux autres doctorants, dont la camaraderie et la solidarité ont été essentielles tout au long de cette thèse : Aaraona, Abdelhak, Antoine, Assaf, Barbara, Benjamin, Bohdan, Clément, Clément, Côme, Cyril, Enzo, Fabio, Hiroshi, Houzhi, Junchao, Laure, Luca, Lucas, Marc, Mi-Song, Mohan, Simon, Sylvain, Yann, Yiyang et Ziad. Merci pour cette recherche collective du meilleur dosage entre temps de travail et de pause. Merci également à Nathalie et Valérie, dont la bienveillance et l'efficacité divisent les difficultés administratives par un facteur appréciable.
J'ai une pensée pour les professeurs de mathématiques de ma scolarité, en particulier pour Martine, Philippe et Serge, dont les clubs auront été un espace de découverte et de liberté très formateur.
Depuis le collège, j'ai eu la grande chance de participer à de nombreuses activités mathématiques extra-scolaires, souvent organisées par des bénévoles, en particulier la POFM, le TFJM et ITYM. J'ai pu me confronter à des problèmes stimulants, et faire des rencontres formidables. C'est là que j'ai vraiment pris conscience des dimensions sociale et internationale des mathématiques. Merci à tous ces encadrants et camarades, de France ou d'ailleurs.
Certains loisirs ont été essentiels pour moi durant cette période, en particulier dans l'association Animath, où j'ai pu prendre part à beaucoup d'activités mathématiques pour la jeunesse. Merci également aux compagnons de l'Étrange Boutique (avec ou sans barbe).
Je remercie ma famille et mes amis pour leur soutien et les bons moments passés tout au long de cette thèse gourmande en temps et en énergie. Je salue aussi leur patience face à mes di-

verses tentatives d'expliquer l'objet de mes recherches : ces discussions m'ont beaucoup aidé à penser différement à mon travail, aussi lointaines puissent-elles paraître de celui-ci. Merci à mes ex-colocataires et mes amis Gros aux préoccupations gourmandes tout court.
Enfin, merci Ting-Chia pour tout ce que tu m'apportes chaque jour - tu es une inépuisable source d'encouragement et d'inspiration.

Résumé

Cette thèse est consacrée à l'étude de différents graphes aléatoires, définis par des propriétés locales (comme la distribution des degrés des sommets ou la probabilité que deux sommets donnés soient reliés par une arête), et dont on cherche à déterminer des caractéristiques globales, notamment leur géométrie et le comportement de marches aléatoires. Elle se compose de trois parties indépendantes. À chaque fois, le graphe aléatoire étudié admet un arbre comme limite locale, et une comparaison fine entre l'arbre et le graphe permet de transposer des propriétés du premier au second.

- La première partie (Chapitre 2) porte sur la limite d'échelle d'un graphe aléatoire critique, un modèle de configuration avec des degrés indépendants et distribués selon une même loi de puissance à queue lourde : elle a une variance, mais pas de troisième moment. Il était connu que les plus grandes composantes connexes de ce graphe ont une taille $\Theta(n^a)$ pour un graphe à n sommets, le paramère a dépendant du modèle. On montre que leur structure converge vers une version biaisée d'un arbre aléatoire continu particulier, l'arbre stable, auquel on rajoute un nombre fini de cycles.

- La seconde partie (Chapitre 3) est consacrée au champ libre gaussien sur des graphes aléatoires d-réguliers. On étudie la percolation du champ libre au-dessus d'un niveau fixé. Si on baisse ce niveau en-dessous d'un certain seuil critique, on établit l'émergence avec grande probabilité d'une unique composante connexe géante englobant une proportion positive des sommets, entourée d'ilôts de taille logarithmique, tandis que seuls ces derniers survivent au-dessus du seuil critique. On montre alors que ce grand continent admet de remarquables similitudes avec la composante géante d'un graphe aléatoire célèbre, le modèle d'Erdős-Rényi.

- La troisième partie (Chapitre 4) traite de marches aléatoires sur des relèvements aléatoires ("random lifts") d'un graphe fini quelconque donné. Ces relèvements sont des graphes peu denses mais avec de bonnes propriétés de connectivité et une structure assez régulière, l'arbre associé étant périodique. On prouve que la marche aléatoire sur ces graphes admet un cutoff, c'est-à-dire qu'il y a une transition brusque entre le moment où le marcheur est encore localisé autour de son point de départ, et celui où on a définitivement perdu sa trace.

Les deux dernières parties ont été réalisées à Paris sous la direction de Justin Salez entre 2017 et 2021. La première est issue d'une collaboration avec Christina Goldschmidt (Oxford Statistics), initiée en 2016 lors d'un stage de recherche et poursuivie jusqu'en 2020.

Abstract

This thesis is devoted to the study of different random graphs, defined by local properties (such as the distribution of the degrees of the vertices, or the probability that two given vertices share an edge). We investigate some of their global characteristics, in particular their geometry and the behaviour of random walks. It consists of three independent parts. In each of them, the local limit of the random graph is a tree, and a fine comparison between the tree and the graph allows to implement properties of the former on the latter.

- The first part (Chapter 2) is about the scaling limit of a critical random graph, more precisely a configuration model with independent and identically distributed degrees having power-law heavy tail behaviour: there is a variance, but no third moment. It was known that the largest connected components of this graph scale like $\Theta(n^a)$ if the graph is on n vertices, for some model-dependent constant a. We prove that their structure converges to a biased version of a particular random $\mathbb{R}$-tree, the stable tree, to which one adds a finite number of cycles.

- The second part (Chapter 3) is devoted to the Gaussian free field on random d-regular graphs. We study its percolation above a fixed level. If this level is below a certain critical threshold, we establish the emergence with high probability of a unique giant component containing a positive proportion of the vertices, surrounded by islets of logarithmic size, while only the latter survive above the critical threshold. We show that this big continent shares remarkable similarities with the giant component of a famous random graph, the Erdős-Rényi model.

- The third part (Chapter 4) deals with random walks on random lifts of an arbitrary finite base graph. Random lifts are sparse graphs with good connectivity properties and a regular structure, the associated tree being periodic. We prove that the random walk on these graphs admits a cutoff, i.e. there is a brutal transition between the time when it is still localised around its starting point, and the time when we have completely lost its track.

The last two parts have been realised in Paris under the supervision of Justin Salez, between 2017 and 2021. The first part stems from a collaboration with Christina Goldschmidt (Oxford Statistics), started in 2016 during a research internship and continued until 2020.

Contents

Chapter 1

Introduction

For over half a century, a wide variety of random graphs have been studied. They were initially introduced as combinatorial and probabilistic objects. The randomness allows to bypass the difficulty to build explicit instances, when one wants to prove the existence of graphs satisfying a given property (P): it suffices to show that (P) is satisfied with positive probability by a graph chosen in a certain random way. We refer the interested reader to the book of Alon and Spencer [19] for pioneering examples of this probabilistic method, in the works of Szele and Erdős in the 1940s.

Nowadays, many random graphs play a central role in the modelling of real-world networks. Popular examples include epidemic spreads, the Internet, interactions between proteins, social networks, and so on. The survey of Newman [118], published in 2003, gives a taste of this diversity. Many applications have been developed since, so that an exhaustive state-of-the-art review would probably be very difficult.

To this inter-disciplinarity corresponds an intra-disciplinarity in probability: recent works on random graphs mix techniques from combinatorics, stochastic analysis, branching processes or ergodic theory for instance. On March 8, 2021 in the MathSciNet database (`https://mathscinet.ams.org/mathscinet/index.html`), there were 12315 papers containing the words "random graphs" in the review text, 2531 of them dating from 2016 or later. 820 and 209 of them respectively had the MSC category 60G (Stochastic processes) as primary or secondary. For the category 05 (Combinatorics), these numbers were 2738 and 713, for the category 37 (Dynamical systems and ergodic theory), 244 and 78.

Both this wide range of applications in various sciences and these numerous connections between different areas of mathematics make random graphs a rich field of study. This thesis illustrates the latter phenomenon. It consists of three parts on purely theoretical models of sparse random graphs, each related to different perspectives and research communities within probability. They are united however by the interaction of several techniques coming from these environments.

In this Introduction, we present some essential features of sparse random graphs, and set the results of this thesis in that context (Section 1.2 for Chapter 2, Section 1.3 for Chapter 3 and Section 1.4 for Chapter 4).

1.1 Sparse random graphs

Large sparse random graphs. Many probabilistic results on random graphs only hold asymptotically with respect to the "size" of the graph, which is often defined as its number of vertices or edges. In this Introduction, $(G_n)_{n\geq 1}$ will denote a sequence of (possibly random) undirected graphs, with vertex set V_n and edge set E_n, and such that $|V_n| = n$. This sequence is said to be **sparse** if $|E_n| = o(n^2)$. In other words, only a vanishing proportion of the potential $n(n-1)/2$ edges exist. In this Introduction, all graphs will satisfy $|E_n| = \Theta(n)$.
One fundamental example of sparse random graph that will illustrate several of the points below is the **Erdős-Rényi random graph**, simultaneously introduced by Gilbert [79] and Erdős and Rényi [75] in 1959. For $n \geq 1$ and $c \in [0, n]$, $\mathrm{ER}(n, c/n)$ is the graph on n vertices such that for every pair of vertices x, y, there is an edge between x and y with probability c/n, independently of all other pairs of vertices. We set c to be a positive constant: the average number of edges is $c/n \times n(n-1)/2 = c(n-1)/2 = \Theta(n)$ so that by the law of large numbers, $\mathrm{ER}(n, c/n)$ is sparse with high probability.

From local rules to global behaviour. Many models are defined by the local interactions between vertices. One knows typically the probability that two vertices share an edge, as in the Erdős-Rényi graph, or the distribution of the degrees of the vertices. A generic example in this case is the **configuration model**: one assigns (in a deterministic or random way) a degree d_i to the i-th vertex of V_n, for every $1 \leq i \leq n$. Then G_n is sampled uniformly at random among all graphs with this degree distribution (this requires that the sum of the d_i's be even, one can for instance replace d_n by $d_n + 1$ if this is not the case).
Then, one tries to infer properties on the global behaviour of G_n as $n \to +\infty$. The questions investigated are often among the following.

- Size: what is the size of the largest of connected components? For instance, what is the probability that G_n is connected as $n \to +\infty$?
- Connectivity: what are the diameter or the typical distance between vertices? How many edges should one remove to disconnect a component?
- Geometry: consider a connected component as a metric space embedded with the standard graph distance (the distance between two vertices being the length of the shortest path), rescaled by the typical distance. Does its distribution converge to some (compact) metric space? If yes, in which metric(s) does the convergence hold?
- Random walks: at what speed does the distribution of a random walker converge to the invariant distribution on a connected component of G_n?

Exploration and comparison with a branching process. A natural way to infer these global properties is to explore the graph neighbour by neighbour, using the random local construction rules. To keep track of this exploration, one records some statistics, for instance the sum of the degrees of the first k vertices visited, $k = 1, 2, \dots$ Due to the sparsity, it is frequently the case that there are few or even no cycles on large parts of the graph. Moreover, the construction rules often entail that each exploration step is almost independent of the previous ones, or only depends on a simple parameter such as the number of vertices already discovered. Hence the part of the graph that we explore can be compared to a branching process T. The latter turns out to be the **local limit** of G_n, i.e. if x is a uniform vertex of G_n and $k \in \mathbb{N}$ an arbitrary radius, the distribution of the ball $B(x, k)$ converges to that of $B(T, k)$. See [59] for a very accessible lecture on this crucial notion. For instance, it is known that the local limit of $\mathrm{ER}(n, c/n)$ is a Galton-Watson tree with a Poisson reproduction law of parameter c.

1.2 The phase transition

Phase transition. Changing some parameters of the model can lead to spectacular modifications of the global picture of G_n. There is often an abrupt phase transition, depending on whether a vertex discovered in the exploration leads in average to more or less than one new neighbour. Generally, if this average is below 1, a typical component that we explore will be close to a subcritical branching process, for instance a subcritical Galton-Watson tree, whose size has exponential moments. Hence, the largest connected components of the graph encompass only $O(\log n)$ vertices. This is the subcritical phase.
On the contrary, if we see more that one new neighbour in average, a connected component of G_n has a positive chance to grow exponentially, until a positive proportion of the vertices have been discovered (which substantially affects the law of the number of new neighbours of each vertex). Then, there is a unique "giant" component of size $\Theta(n)$, with good connectivity properties. This is the supercritical phase. The limit case, when this number is close to 1, is generally harder to characterize. The possible existence and properties of a critical regime that would drastically differ from both the subcritical and supercritical ones usually offer thrilling research directions.
For instance, Erdős and Rényi [74] showed in 1960 that $\mathrm{ER}(n, c/n)$ undergoes such a phase transition. The supercritical regime corresponds to $c > 1$ and the subcritical regime to $c < 1$.

Random graphs at criticality. When the average number of new neighbours is 1, we might expect that the part of the graph that we explore is close to a critical branching process.
In the critical phase $c = 1$, Aldous [13] established in a celebrated paper in 1997 that the largest connected components of $\mathrm{ER}(n, 1/n)$, whose number grows to infinity with n, are of size $\Theta(n^{2/3})$. To do so, he built an exploration process, that roughly counts the number of vertices "seen" (i.e. we have explored one of their neighbours) but not yet explored. It converges to a slightly modified Brownian motion when suitably rescaled. This exponent $2/3$ comes in fact

from the fractal invariance of the Brownian motion: after $n^{2/3}$ vertices have been seen, each discovered vertex leads in average to $(n - n^{2/3})/n = 1 - n^{-1/3}$ new vertices. If we want a connected component to survive, we should discover at least 1 new vertex at each explored vertex, so that there is a "deficit" of order $n^{2/3} \times n^{-1/3} = n^{1/3}$ vertices, that can be compensated by the Brownian fluctuations of order $(n^{2/3})^{1/2} = n^{1/3}$ of the stack of vertices seen but not yet explored.
Fifteen years later, Addario-Berry, Broutin and Goldschmidt [6] found a scaling limit for the structure of these largest components: when scaling the distances by a factor $n^{-1/3}$, they converge to a sequence of random trees, with a finite number of identifications of pairs of vertices to create cycles. These random trees are modifications of the Continuum Random Tree (CRT), a random tree which is coded by an excursion of the Brownian motion above 0. The CRT is an unavoidable bridge between random graphs and stochastic calculus, and has been thoroughly studied since the 1990s (see [10, 11, 12, 95]).

Universal scaling limits. This phase transition has been investigated on various other random graphs: configuration models, inhomogeneous random graphs, preferential attachment models, and so on (see for instance [36, 50, 61, 76, 115], we give a detailed bibliographical account in Section 2.2.5 of Chapter 2). The book of van der Hofstad [135] gives a modern panorama of these models. It turns out that in most cases, one of the following two scenarii happens:

(I) the degree distribution of a vertex $x \in V_n$ chosen uniformly at random has a finite third moment. Then the largest connected components have size $\Theta(n^{2/3})$, the diameter and typical distances between vertices are $\Theta(n^{1/3})$.

(II) this degree distribution has a finite second moment but its third moment is infinite. If $\mathbb{P}(\deg(x) = k) \underset{k\to+\infty}{\sim} L(k)k^{-(\alpha+2)}$ for some $\alpha \in (1, 2)$ and some slowly varying function L (for simplicity, think of it as a positive constant), then the largest connected components have size $\Theta(n^{\alpha/(\alpha+1)})$. The diameter and typical distances between vertices are $\Theta(n^{(\alpha-1)/(\alpha+1)})$.

When the degree distribution has infinite variance, there are enough "hubs" (vertices with arbitrarily large degrees as $n \to +\infty$) so that all pairs of hubs are connected, and every vertex is connected to a hub, hence distances in G_n are drastically shrunk.
In both (I) and (II), one usually shows that the exploration process, when suitably rescaled, converges to a (possibly modified) Lévy process. The case (II) is often the hardest from a technical point of view: there are many large degrees that make some key functionals non-integrable but not enough to simplify the structure of G_n as in the infinite variance case.
In Chapter 2, we study a natural generic model that belongs to this case. It is a critical configuration model with i.i.d. degrees whose distribution ν satisfies (II). We prove that the rescaled largest connected components converge to modifications of a random tree coded by an

α-stable Lévy process, the α-stable tree, with a finite number of vertex identifications to create cycles (Theorem 2.1.1). These limiting spaces are new objects for $\alpha \in (1, 2)$, whose study has recently been initiated in [82]. This convergence generalizes what happens in the special case $\alpha = 2$ (which belongs to (I) - the CRT is the α-stable tree for $\alpha = 2$).
The originality and interest of our method lies in a measure-change that gives the law of the degrees met in the exploration w.r.t. a sequence of i.i.d. degrees distributed as $\widetilde{\nu}$, the size-bias of ν. It quantifies naturally an intuitive phenomenon, the progressive "degree depletion" of the vertices met during the exploration. In the latter, each new vertex is picked with a probability proportional to its degree, hence according to the size-bias of the distribution of the remaining degrees. But $\widetilde{\nu}$ dominates stochastically ν, so that we tend to visit the vertices with a larger degree first. This measure-change converges in the limit, to express how our limiting spaces are obtained from stable trees. It would be interesting to look for a generalization of this procedure to other graphs.
Contrary to many works on similar models, our approach is independent from the multiplicative coalescent. The latter dynamic, already studied by Aldous in his seminal paper [9], is suited to models with a monotonicity w.r.t. their parameters, like $\mathrm{ER}(n, p/n)$: edges are progressively added as one raises the value of p, and the probability that a link appears between two connected components is approximately proportional to the product of their sizes.

1.3 The Gaussian Free Field

The Gaussian Free Field on transient graphs. On an infinite transient graph G, one can define a centred Gaussian process $(\varphi(x))_{x \in G}$ indexed by its vertices, such that its covariances are given by the Green function Γ_G: for two vertices x, y of G,

$$\mathrm{Cov}(\varphi(x), \varphi(y)) = \Gamma_G(x, y) = \sum_{k \geq 0} \mathbb{P}(X_k = y),$$

where $(X_k)_{k \geq 0}$ is a Simple Random Walk (SRW) on G, starting at x. This process is called the **Gaussian Free Field** (GFF) on G. By this construction involving the Green function, the GFF is deeply connected to structural properties of G and to the behaviour of SRWs on G. In the past two decades, several generalizations of the second Ray-Knight theorem have been found [73, 97, 104, 125], linking the distribution of the GFF with that of the local times of the random walk on G. In the same spirit, connections with random interlacements have been established [130, 131].
One way to study the GFF is to look at its level-sets. For $h \in \mathbb{R}$, the **level-set above** h of φ, denoted $E^{\geq h}$, is the subgraph of G induced by the set of vertices $\{x, \varphi(x) \geq h\}$. Level-set percolation of the GFF has been investigated since the 1980s [48, 111]. The last decade has witnessed blossoming literature on classical graphs, in particular $\mathbb{Z}^d$ ([65, 68, 104, 121, 123]) and transient trees ([3, 130, 132]).

One central question is whether there exists (with positive probability) an infinite connected component in $E^{\geq h}$, depending on h. In particular, is there a phase transition, i.e. does a non-degenerate threshold $h_G \in \mathbb{R}$ exist, such that the answer is yes for $h < h_G$ and no for $h > h_G$? Compared to classical percolation models, such as Bernoulli bond percolation where all edges are independent, the GFF level-set percolation brings long-range correlations that may play an important role, in particular in graphs like $\mathbb{Z}^d$ where the Green function decays only polynomially w.r.t. the distance between vertices. Very recently, GFF percolation has even been used to show a result on Bernoulli bond percolation [67].

GFF level-sets on finite graphs. If G_n is connected, one can define an analogous field on G_n, the **zero-average Gaussian Free Field**. Its covariance is given by the zero-average Green function Γ_{G_n}: for $x, y \in V_n$,

$$\Gamma_{G_n}(x, y) := \int_0^{+\infty} (\mathbb{P}(\overline{X}_t = y) - \pi_n(y))\, dt,$$

where $(\overline{X}_t)_{t\geq 0}$ is a continuous-time SRW on G_n: its trajectory is that of a SRW, and the time intervals between two consecutive jumps are i.i.d. with law Exp(1). π_n is the invariant distribution of the SRW on G_n.

If $(G_n)_{n\geq 0}$ has a local limit G, a natural question is whether some characteristics of the GFF on G can be transferred to the zero-average GFF on G_n. For instance, one might ask whether a phase transition for the existence of an infinite connected component of the level-set $E^{\geq h}$ in G corresponds to a phase transition for the emergence of a "macroscopic" component of size $\Theta(n)$ in the level-set $E_n^{\geq h}$ above h in G_n.

For $G = \mathbb{T}_d$ the d-regular tree, Sznitman [132] showed that there exists a level-set percolation threshold $h_\star \in (0, \infty)$. Abächerli and Černý [4] proved that if G_n is a typical d-regular graph, then for $h > h_\star$ (subcritical phase), the connected components of $E_n^{\geq h}$ have size $O(\log n)$, and for $h < h_\star$ (supercritical phase), a positive proportion of the vertices are in mesoscopic components of size at least $\Theta(n^a)$, for some non explicit constant $a > 0$. The question on the existence of a macroscopic component (i.e. $a = 1$) was left open.

In Chapter 3, we give a positive answer for most regular graphs: if G_n is a uniform random d-regular graph, then w.h.p. $E_n^{\geq h}$ has a unique giant component $\mathcal{C}_1^{(n)}$ of size $\Theta(n)$. More precisely, $|\mathcal{C}_1^{(n)}| = \eta(h)n(1 + o(1))$, where $\eta(h) > 0$ is the probability that the connected component $\mathcal{C}^h$ of a given vertex of $\mathbb{T}_d$ in the level-set above h is infinite (Theorem 3.1.1).

We also show that $\mathcal{C}_1^{(n)}$ shares several "global" properties with the giant component of ER$(n, p/n)$ for $p > 1$ (hence in the supercritical regime), concerning the diameter, the typical distance between vertices, the core and the kernel (Theorem 3.1.2). The local limit of $\mathcal{C}_1^{(n)}$ is $\mathcal{C}^h$ conditioned to be infinite.

Our proofs rely on an annealed exploration of a new kind, in which we reveal progressively the structure of G_n and the zero-average GFF on it. We also need to refine some properties of

the component $\mathcal{C}^h$ in $\mathbb{T}_d$ shown in [3]. In a work in progress [57], we study closely $\mathcal{C}^h$ and the random walk on it, proving its ballisticity.
It would be exciting to determine what happens when $h = h_\star$: are we in (I) or (II), or does the GFF level-set percolation lead to a new universality class of critical random graphs?

1.4 The mixing time

Mixing time and cutoff. A natural way to explore a graph and record information on its structure is to study random walks on its vertices. A quantity of great interest is the mixing time, i.e. the time it takes for a random walk to be approximately distributed according to the invariant probability measure on the graph. Formally, let P_n be the transition matrix of the discrete-time Simple Random Walk (SRW) on V_n. Suppose that G_n is connected, so that there is a unique invariant distribution π_n. For $\varepsilon > 0$ and for a vertex x of G_n, the **ε-mixing time from x** is

$$t_x^{(n)}(\varepsilon) := \inf\{t \in \mathbb{N} \cup \{0\},\ \|P_n^t(x, \cdot) - \pi_n\|_{TV} \leq \varepsilon\}.$$

One often examines the **worst-case mixing time** $t_{max}^{(n)}(\varepsilon) := \sup\{t_x^{(n)}(\varepsilon),\ x \in V_n\}$. Other distances than the total variation distance might be considered.
The mixing time delivers essential information on the geometry of the graph. For instance, it provides a lower bound for the typical distance between vertices. If the latter is much smaller than the mixing time, then the graph may have a weak expansion, or traps that delay the random walk. If the 1/10-mixing time is much larger than the 1/2-mixing time, then there might be a bottleneck in the graph that is hard to cross for the random walk, and so on.
The mixing time also plays a crucial role in Monte Carlo simulations, as it gives a good estimate for their running time. The books of Levin, Peres and Wilmer [98] and of Montenegro and Tetali [114] give a broad panorama of both historical and modern techniques on mixing times.
We say that there is **cutoff** when for all $\varepsilon, \varepsilon' \in (0, 1)$,

$$t_{max}^{(n)}(\varepsilon)/t_{max}^{(n)}(\varepsilon') \to 1 \text{ as } n \to +\infty.$$

In other words, the cutoff means that the mixing of the random walk happens abruptly. First examples of this phenomenon were given in the 1980s for random walks on finite groups (see [14] or [63] on the symmetric group) or on spaces that can be factored into a n-product of a base space (such as $\mathbb{Z}_2^n$ in [9]). This "algebraic" direction is still investigated nowadays. See for instance [83, 85] on random Cayley graphs of abelian groups.

Expanders. A wide class of graphs where random walks mix fast and cutoff might happen are the expander graphs. We say that the sequence $(G_n)_{n\geq 1}$ is **expanding** if its isoperimetric constant is bounded away from 0, i.e. there exists $c > 0$ such that for large enough n,

$$c_n := \inf_{S \subseteq V_n, |S| \leq n/2} \frac{|\partial S|}{|S|} \geq c,$$

where $\partial S \subseteq V_n \setminus S$ is the set of vertices having one neighbour in S. The very accessible survey of Hoory, Linial and Widgerson [87] provides a good overview of the study of these graphs. Some explicit constructions of expanders have been achieved since the 1980s, using number theory and combinatorial arguments. The book of Lubotzky [103] gives a good account on it. Finding such constructions is a difficult task. It is much easier to prove the existence of expanders via a probabilistic argument. If G_n is a d-regular graph on n vertices (i.e. all vertices have degree d) chosen uniformly at random, then a combinatorial computation shows that with high probability, G_n is an expander.

This expansion property is equivalent to the existence of a spectral gap: let λ_n be the second largest eigenvalue of P_n (the largest is 1, P_n being a stochastic matrix). Then $\liminf_{n\to+\infty} c_n > 0$ if and only if $\liminf_{n\to+\infty} \lambda_n > 0$. The implication (expansion $\Rightarrow$ spectral gap) was first formulated by Cheeger [53] in 1969, for Riemannian manifolds. To our knowledge, the earliest adaptation to random walks on discrete graphs is due to Alon and Milman [17] in 1984. It is classical that a spectral gap bounded away from 0 implies in turn that the SRW on G_n mixes in $O(\log n)$ steps, provided we make it lazy, i.e. at each step, the random walker flips a balanced coin and stays on its position if and only if the result is heads. Conversely, if the mean degree of a vertex of G_n is bounded, and the degree distribution is sufficiently regular, one might expect that the SRW does not mix in less than $\Theta(\log n)$ steps. Therefore, the mixing should happen after $\Theta(\log n)$ steps.

Cutoff on sparse expanders. A major breakthrough came in 2010, when Lubetzky and Sly proved that the SRW on the random d-regular graph has a cutoff [101].

Theorem [Lubetzky & Sly, 2010] *If G_n is a uniform d-regular random graph, then for every $\varepsilon \in (0,1)$,*

$$\frac{t_{max}^{(n)}(\varepsilon) - \frac{d}{d-2}\log_{d-1} n}{\sqrt{\log_{d-1} n}} \xrightarrow{\mathbb{P}} \frac{2\sqrt{d(d-1)}}{(d-2)^{3/2}}\Phi^{-1}(\varepsilon),$$

where $\Phi(x) := \int_x^{+\infty}(2\pi)^{-1/2}\exp(-u^2/2)du$ for $x \in \mathbb{R}$ is the tail distribution of the standard normal.

Rather than a simple asymptotic on the mixing time, this result is very similar to a central limit theorem, with a precise cutoff window $\Theta(\sqrt{\log n})$. In fact, it reflects the behaviour of the entropy of the SRW on $\mathbb{T}_d$. If $P_{(d)}$ is the transition matrix of the SRW on $\mathbb{T}_d$, then $t^{-1/2}(\log_{d-1} P_{(d)}^t(X_0, X_t) - \frac{d-2}{d}t)$ converges in distribution to a centred Gaussian variable as $t \to +\infty$. Then, by counting arguments limiting w.h.p. the occurrence of cycles around the trajectory of the SRW on G_n, one can show that $\log_{d-1} P_{(d)}^t(X_0, X_t)$ and $\log_{d-1} P_n^t(X_0, X_t)$ are close. The mixing happens when $P_n^t(X_0, X_t) \simeq 1/n$, and this occurs with positive probability for $t = \frac{d}{d-2}\log_{d-1} n + \Theta(\sqrt{\log_{d-1} n})$.

Since then, several results were proved on similar models, following roughly this pattern: cutoff happens at an entropic time, the constant being given by a law of large numbers for the entropy of the random walk on the local limit of $(G_n)_{n\geq 1}$. In the same article, Lubetzky and Sly also showed a cutoff for the Non-Backtracking Random Walk (NBRW, i.e. a SRW conditioned at each step on not going back along the edge it has just crossed). See also [29] for the SRW on the largest component of a supercritical Erdős-Rényi random graph, or [27] for the NBRW on a configuration model. Very recently, Hermon, Sly and Sousi [84] found a generic way to produce cutoff by adding new edges to a deterministic graph G_n (on which one makes mild assumptions), via a perfect matching of its vertices.

Over the last few years, there has been increasing interest in mixing times on dynamical graphs (typically, edges are re-sampled at random at a given rate), when the mixing time profile is already well-known on a static version of the graph, see [24, 51, 128].

In Chapter 4, we study the random walk on random lifts of a fixed graph G, which is a natural way to combine the "product of a base space" and the "expanding sparse graph" perspectives for cutoff. To obtain a uniform random n-lift of G, take n copies of the vertices of G, and for every pair of vertices u, v sharing an edge e in G, draw an edge from the i-th copy of u to the $\sigma_e(i)$-th copy of v, the σ_e's being independent uniform random permutations of $\{1, \ldots, n\}$.

Our main result is Theorem 4.1.1: we prove the cutoff for the random walk at entropic time, with a cutoff window $\Theta(\sqrt{\log n})$ (with an explicit Gaussian profile only for the lower bound). We make very few assumptions on the base graph G. In particular, we allow weights on the edges that make the random walk non-reversible. The proof relies on a fine understanding of the random walk on the universal cover of G, a periodic tree T_G, which is the local limit of G_n. Then, a careful exploration, where the structure of G_n is revealed along the trajectory of a random walk, allows to transpose properties from T_G to G_n. A byproduct of our method is the computation of the diameter of random lifts (Theorem 4.3.1).

In addition, we prove that the NBRW on non-weighted random lifts has a cutoff, with a Gaussian profile for the cutoff window (Theorem 4.2.3).

Chapter 2

Scaling limit of a critical configuration model with power-law degrees

This chapter stems from the preprint [58], that has been submitted to Annals of Probability and accepted with major revisions.

Abstract. We prove a metric space scaling limit for a critical random graph with independent and identically distributed degrees having power-law tail behaviour with exponent $\alpha+1$, where $\alpha \in (1,2)$. The limiting components are constructed from random $\mathbb{R}$-trees encoded by the excursions above its running infimum of a process whose law is locally absolutely continuous with respect to that of a spectrally positive α-stable Lévy process. These spanning $\mathbb{R}$-trees are measure-changed α-stable trees. In each such $\mathbb{R}$-tree, we make a random number of vertex-identifications, whose locations are determined by an auxiliary Poisson process. This generalises results which were already known in the case where the degree distribution has a finite third moment (a model which lies in the same universality class as the Erdős–Rényi random graph) and where the role of the α-stable Lévy process is played by a Brownian motion.

Contents

2.1 Introduction

2.1.1 Overview

In recent years, a wide variety of random graph models have been introduced and studied. Many of these models undergo a phase transition of the following type: below some threshold, the connected components are microscopic in size (in the sense that they each contain a negligible proportion of the vertices) and possess few cycles, whereas above the threshold, there is a component which occupies a positive fraction of the vertices and contains many cycles, and all other components are again microscopic. We are particularly interested in the behaviour exactly at the point of the phase transition, and in a precise description of the sizes and geometric properties of the components, which is typically much more delicate than in the sub- and supercritical cases. We will first give a brief overview of the setting in which we are interested, and of our main results, deferring a more detailed account with proper definitions, as well as a summary of the pre-existing literature, to the next section.

We consider a uniform random graph on n vertices with a given degree sequence, where the degrees themselves are independent and identically distributed random variables, $D_1, \ldots, D_n$. (If $\sum_{i=1}^{n} D_i$ is odd, we replace D_n by $D_n + 1$.) For simplicity, we impose the condition that $\mathbb{P}(D_1 \geq 1) = 1$, so that there are no isolated vertices. We also assume that $\mathbb{P}(D_1 = 2) < 1$ (otherwise we have a random 2-regular graph, which contains many cycles of macroscropic size [23]) and that $\operatorname{var}(D_1) < \infty$ (otherwise the graph behaves very differently; see [129]). The phase transition then occurs when the parameter $\theta := \mathbb{E}[D_1(D_1 - 1)] / \mathbb{E}[D_1]$ passes through 1: if $\theta < 1$ then the proportion of vertices in the largest component tends to 0 in probability as $n \to \infty$, whereas if $\theta > 1$, this proportion instead converges to a strictly positive constant, again in probability as $n \to \infty$.

At the critical point $\theta = 1$, there is a sequence of components whose sizes are comparable (rather than a single giant component, as in the supercritical case) and which, even after rescaling, retain some randomness in the limit. The sizes and geometric properties of these components depend on the tail of the distribution of D_1. In particular,

- if $\mathbb{E}\left[D_1^3\right] < \infty$ then the largest components have sizes on the order of $n^{2/3}$ and diameters on the order of $n^{1/3}$;

- if $\mathbb{P}(D_1 = k) \sim ck^{-(\alpha+2)}$ for some constant $c > 0$ and $\alpha \in (1, 2)$ then the largest components have sizes on the order of $n^{\alpha/(\alpha+1)}$ and diameters on the order of $n^{(\alpha-1)/(\alpha+1)}$.

(It will be convenient to refer to the first of these as the "$\alpha = 2$ case".)

These scaling properties are either proved or conjectured to be *universal*, that is to hold for whole families of random graph models with similar asymptotic degree distributions. We will discuss the issue of universality in some detail below.

2.1.2 Our results

Let us now state our scaling limit theorem. Let $G_1^n, G_2^n, \ldots$ be the (vertex-sets of the) components of the critical random graph, listed in decreasing order of size, with ties broken arbitrarily. We think of these as measured metric spaces, by endowing G_i^n with the graph distance, d_i^n, and the counting measure on its vertices, μ_i^n. Formally, each is an element of the Polish space of isometry-equivalence classes of measured metric spaces endowed with the Gromov–Hausdorff–Prokhorov distance, which we will define properly below.

Theorem 2.1.1. *Fix $\alpha \in (1, 2]$. Then under the conditions above, there exists a sequence of random compact measured metric spaces $(\mathcal{G}_1, d_1, \mu_1), (\mathcal{G}_2, d_2, \mu_2), \ldots$ (whose law depends on α) such that, as $n \to \infty$,*

$$\left(\left(G_i^n, n^{-(\alpha-1)/(\alpha+1)} d_i^n, n^{-\alpha/(\alpha+1)} \mu_i^n\right), i \geq 1\right) \xrightarrow{d} \left((\mathcal{G}_i, d_i, \mu_i), i \geq 1\right)$$

in the sense of the product Gromov–Hausdorff–Prokhorov topology.

The same result also holds for a multigraph with the same degree sequence generated according to the configuration model (see Section 2.2.1 for more details).

In the terminology of [7], $(\mathcal{G}_i, d_i)$ is a random $\mathbb{R}$-graph for each $i \geq 1$. For reasons which will shortly become clear, we refer to the whole limiting object as the *α-stable graph* if $\alpha \in (1, 2)$ or the *Brownian graph* (instances of which have already occurred several times in the literature) if $\alpha = 2$.

This theorem, in particular, implies the scaling properties mentioned above. The $\alpha = 2$ case may be deduced from a more general theorem due to Bhamidi and Sen [37], proved by different methods. For $\alpha \in (1, 2)$, Bhamidi, Dhara, van der Hofstad and Sen [36] considered the setting of critical percolation on a *supercritical* uniform random graph with given degree sequence, having similar tail behaviour to ours, and proved a scaling limit theorem in the sense of the product Gromov-weak topology. (We understand that this will be improved to a convergence in the product Gromov–Hausdorff–Prokhorov topology for *critical* degree sequences satisfying certain conditions in forthcoming work [35].) We will describe the results of [36] in more detail below and will, for the moment, simply observe that there are situations which are covered by both theorems, and where the limit objects must therefore be the same, but where this is certainly not obvious from their respective constructions.

One of the most striking aspects of our results is the characterisation of the limit spaces which we are able to give, which is completely new for $\alpha \in (1, 2)$, and generalises one which was already known for $\alpha = 2$. In order to give this characterisation, we must first introduce some stochastic processes which play a key role.

Let $\mu = \mathbb{E}[D_1]$. For $\alpha \in (1, 2)$, let L be a spectrally positive α-stable Lévy process with Laplace transform

$$\mathbb{E}[\exp(-\lambda L_t)] = \exp\left(\frac{c\Gamma(2-\alpha)}{\mu\alpha(\alpha-1)}\lambda^\alpha t\right), \quad \lambda \geq 0, \quad t \geq 0,$$

where $c > 0$ is the constant such that $\mathbb{P}(D_1 = k) \sim ck^{-(\alpha+2)}$. Such a process can be thought of as encoding a forest of continuum trees; the standard way to do this goes via a (somewhat complicated) functional of L called the height process H (we will define this properly below). Let

$$C_\alpha = \frac{c\Gamma(2-\alpha)}{\alpha(\alpha-1)}.$$

We will create a new pair $(\widetilde{L}, \widetilde{H})$ of processes via change of measure as follows: for suitable test-functions $f : \mathbb{D}([0,t], \mathbb{R})^2 \to \mathbb{R}$, let

$$\mathbb{E}\left[f(\widetilde{L}_u, \widetilde{H}_u, 0 \leq u \leq t)\right] = \mathbb{E}\left[\exp\left(-\frac{1}{\mu}\int_0^t s dL_s - \frac{C_\alpha t^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right) f(L_u, H_u, 0 \leq u \leq t)\right]. \tag{2.1}$$

For $\alpha = 2$, letting $\mu = \mathbb{E}[D_1]$ and $\beta = \mathbb{E}[D_1(D_1-1)(D_1-2)]$, we instead take

$$L_t = \sqrt{\frac{\beta}{\mu}} B_t \quad \text{and} \quad H_t = 2\sqrt{\frac{\mu}{\beta}}\left(B_t - \inf_{0 \leq s \leq t} B_s\right),$$

where B is a standard Brownian motion (in the Brownian setting, the associated height process has the same distribution as a reflected Brownian motion, up to a scaling constant). In this case, define

$$\mathbb{E}\left[f(\widetilde{L}_u, \widetilde{H}_u, 0 \leq u \leq t)\right] = \mathbb{E}\left[\exp\left(-\frac{1}{\mu}\int_0^t s dL_s - \frac{\beta t^3}{6\mu^3}\right) f(L_u, H_u, 0 \leq u \leq t)\right]. \tag{2.2}$$

In either case, let

$$R_t = \widetilde{L}_t - \inf_{0 \leq s \leq t} \widetilde{L}_s, \quad t \geq 0.$$

Now write $(\zeta_i, i \geq 1)$ for the ordered sequence of lengths of excursions of R above 0. These excursions give rise to spanning $\mathbb{R}$-trees for the limiting components. For $i \geq 1$, let $\widetilde{\varepsilon}_i : [0, \zeta_i] \to \mathbb{R}_+$ be the ith longest excursion of R (with its argument translated in the natural way to $[0, \zeta_i]$). For $i \geq 1$, let $\widetilde{h}_i : [0, \zeta_i] \to \mathbb{R}_+$ be the corresponding (continuous) excursion of H above 0 (which has the same length as $\widetilde{\varepsilon}_i$). Let $(\widetilde{\mathcal{T}}_1, \widetilde{d}_1, \widetilde{\mu}_1), (\widetilde{\mathcal{T}}_2, \widetilde{d}_2, \widetilde{\mu}_2), \ldots$ be the measured $\mathbb{R}$-trees encoded by $\widetilde{h}_1, \widetilde{h}_2, \ldots$ respectively, and write p_i for the canonical projection from $[0, \zeta_i]$ to $\widetilde{\mathcal{T}}_i$, for $i \geq 1$.

Conditionally on R, now consider a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}_+$ of intensity $\frac{1}{\mu} \mathbb{1}_{\{x \le R_t\}} dt dx$. (Equivalently, we can think of this as a Poisson point process of intensity $1/\mu$ in the area under the graph of R.) The points tell us how to identify vertices in the $\mathbb{R}$-trees in order to create cycles. For $i \ge 1$, suppose that a number $M_i \ge 0$ of points fall within the ith longest excursion $\widetilde{\varepsilon}_i$. Given $\widetilde{\varepsilon}_i$, we then have $M_i \sim \text{Poisson}\left(\frac{1}{\mu} \int_0^\infty \widetilde{\varepsilon}_i(u) du\right)$. If $M_i \ge 1$, write

$$(s_{i,1}, x_{i,1}), (s_{i,2}, x_{i,2}), \ldots, (s_{i,M_i}, x_{i,M_i})$$

for the points themselves (with their first co-ordinates translated to the interval $[0, \zeta_i]$). For $i \ge 1$ and $1 \le k \le M_i$, let

$$t_{i,k} = \inf\{t \ge s_{i,k} : \widetilde{\varepsilon}_i(t) \le x_{i,k}\}.$$

Now for $i \ge 1$, let $(\mathcal{G}_i, d_i, \mu_i)$ be the measured metric space obtained from $(\widetilde{\mathcal{T}}_i, \widetilde{d}_i, \widetilde{\mu}_i)$ by making no change if $M_i = 0$ or, if $M_i \ge 1$, by identifying the M_i pairs of points

$$(p_i(s_{i,1}), p_i(t_{i,1})), \ldots, (p_i(s_{i,M_i}), p_i(t_{i,M_i})).$$

(Formally, this is done by taking the quotient metric space in a standard way which is described in detail, for example, just before Lemma 21 of [6].)

Conditionally on the ordered lengths $\zeta_1, \zeta_2, \ldots$ of the excursions and numbers $M_1, M_2, \ldots$ of Poisson points, we may give an attractive alternative description of the excursions encoding the spanning forests of the α-stable and Brownian graphs. These are closely related to the canonical family of random $\mathbb{R}$-trees encompassing the Brownian continuum random tree [10, 11, 12] and α-stable trees [70, 69], which are the scaling limits of critical Galton–Watson trees conditioned to have size n with offspring distribution in the domain of attraction of a Normal or α-stable distribution respectively.

First consider $\alpha \in (1, 2)$, and let $\mathbb{e}$ be a normalised (i.e. length 1) excursion of the stable process L, and let $\mathbb{h}$ be the associated normalised excursion of the height process, which would encode an α-stable tree. Now for $m \in \mathbb{Z}_+$, define tilted excursions $\widetilde{\mathbb{e}}^{(m)}$ and $\widetilde{\mathbb{h}}^{(m)}$ via

$$\mathbb{E}\left[g(\widetilde{\mathbb{e}}^{(m)}, \widetilde{\mathbb{h}}^{(m)})\right] = \frac{\mathbb{E}\left[g(\mathbb{e}, \mathbb{h}) \left(\int_0^1 \mathbb{e}(u) du\right)^m\right]}{\mathbb{E}\left[\left(\int_0^1 \mathbb{e}(u) du\right)^m\right]}, \tag{2.3}$$

for suitable test-functions $g : \mathbb{D}([0,1], \mathbb{R}_+) \times \mathbb{C}([0,1], \mathbb{R}_+) \to \mathbb{R}$. Let $(\widetilde{\mathcal{T}}^{(m)}, \widetilde{d}^{(m)}, \widetilde{\mu}^{(m)})$ be the $\mathbb{R}$-tree $(\widetilde{\mathcal{T}}^{(m)}, \widetilde{d}^{(m)})$ encoded by $\widetilde{\mathbb{h}}^{(m)}$, along with its natural mass measure $\widetilde{\mu}^{(m)}$. Write $\widetilde{p}^{(m)}$ for the projection $[0,1] \to \widetilde{\mathcal{T}}^{(m)}$. If $m \ge 1$, now sample m pairs of points in the tree as follows. First pick $(s_1, x_1), \ldots, (s_m, x_m)$ independently and uniformly from the area below the excursion $\widetilde{\mathbb{e}}^{(m)}$ and above the x-axis according to the normalised Lebesgue measure. Define $t_i = \inf\{t \ge s_i : \widetilde{\mathbb{e}}^{(m)}(t) \le x_i\}$. Finally, identify $\widetilde{p}^{(m)}(s_i)$ and $\widetilde{p}^{(m)}(t_i)$ for $1 \le i \le m$ in order to obtain $(\mathcal{G}^{(m)}, d^{(m)}, \mu^{(m)})$. Set $(\mathcal{G}^{(0)}, d^{(0)}, \mu^{(0)}) = (\widetilde{\mathcal{T}}^{(0)}, \widetilde{d}^{(0)}, \widetilde{\mu}^{(0)})$.

Something very similar works in the Brownian case. Here, we take $\mathbb{e}$ to be a normalised Brownian excursion (which is, in particular, continuous); in this context, $\mathbb{h} = 2\mathbb{e}$, so there is no need to consider two different excursions. The function $2\mathbb{e}$ encodes the Brownian continuum random tree (in the normalisation adopted by Aldous [10]). Again define $\tilde{\mathbb{e}}^{(m)}$ as at (2.3) and let $(\widetilde{\mathcal{T}}^{(m)}, \tilde{d}^{(m)}, \tilde{\mu}^{(m)})$ be the measured $\mathbb{R}$-tree encoded by $2\tilde{\mathbb{e}}^{(m)}$, and write $\tilde{p}^{(m)}$ for the projection $[0,1] \to \widetilde{\mathcal{T}}^{(m)}$. If $m \geq 1$, now sample m pairs of points in the tree as follows. First pick $(s_1, x_1), \ldots, (s_m, x_m)$ independently and uniformly from the area below the excursion $\tilde{\mathbb{e}}^{(m)}$ and above the x-axis according to the normalised Lebesgue measure. Define $t_i = \inf\{t \geq s_i : \tilde{\mathbb{e}}^{(m)}(t) \leq x_i\}$. Finally, identify $\tilde{p}^{(m)}(s_i)$ and $\tilde{p}^{(m)}(t_i)$ for $1 \leq i \leq m$ in order to obtain $(\mathcal{G}^{(m)}, d^{(m)}, \mu^{(m)})$. Set $(\mathcal{G}^{(0)}, d^{(0)}, \mu^{(0)}) = (\widetilde{\mathcal{T}}^{(0)}, \tilde{d}^{(0)}, \tilde{\mu}^{(0)})$.

Theorem 2.1.2. *Conditionally on the lengths $\zeta_1, \zeta_2, \ldots$ of the excursions and the numbers $M_1, M_2, \ldots$ of points, the measured $\mathbb{R}$-graphs $(\mathcal{G}_1, d_1, \mu_1)$, $(\mathcal{G}_2, d_2, \mu_2), \ldots$ are independent with*

$$(\mathcal{G}_i, d_i, \mu_i) \stackrel{d}{=} \left(\mathcal{G}^{(M_i)}, \zeta_i^{(\alpha-1)/\alpha} d^{(M_i)}, \zeta_i \mu^{(M_i)}\right)$$

for each $i \geq 1$.

This shows that, in order to understand further the geometric properties of our limit object, a key role will be played by the family of random $\mathbb{R}$-graphs $((\mathcal{G}^{(m)}, d^{(m)}, \mu^{(m)}), m \geq 0)$. These are studied in depth for $\alpha \in (1,2)$ in the companion paper [82]; the Brownian case was the subject of the earlier paper [5]. From the absolute continuity relation (2.3), for any $\alpha \in (1,2]$ it is straightforward to see that $(\mathcal{G}^{(m)}, d^{(m)}, \mu^{(m)})$ has Hausdorff dimension $\alpha/(\alpha-1)$ almost surely, since this is true of the appropriate Brownian/α-stable tree and one cannot change the fractal dimension by making finitely many vertex-identifications.

The branch-points of the α-stable tree are almost surely all infinitary (i.e. removing any of them breaks the tree into infinitely many connected components), and this property is inherited, via absolute continuity, by $(\widetilde{\mathcal{T}}^{(m)}, \tilde{d}^{(m)})$ for $\alpha \in (1,2)$. It follows from the properties of the excursion $\mathbb{e}^{(m)}$ (see [82] for an in-depth discussion) that the vertex-identifications in $(\widetilde{\mathcal{T}}^{(m)}, \tilde{d}^{(m)})$ are almost surely all from a leaf to a branch-point of infinite degree. In contrast, in the $\alpha = 2$ case, the vertex-identifications are almost surely all from a leaf to a point of degree 2 (see [6, 5]).

In [5, 82], it is further shown that one may explicitly determine the law of the *kernel* of $\mathcal{G}^{(m)}$ (that is, the multigraph with edge-lengths which encodes its cycle structure), and that $\mathcal{G}^{(m)}$ may be constructed by gluing together randomly rescaled Brownian/stable trees. Finally, it is shown in [5, 82] that $\mathcal{G}^{(m)}$ possesses a *line-breaking construction*, that is, a recursive construction which starts from the kernel and successively glues on line-segments of random lengths to random points, obtaining a convergent sequence of approximations to the final $\mathbb{R}$-graph.

2.2 Background

In this section, we give some background material on our random graph model, and discuss the previously known results on its critical behaviour. We also give a brief account of the scaling limit theory for Galton–Watson trees. We then give an overview of the proof of Theorem 2.1.1. This is followed by a brief summary of some related literature, and some open problems. Finally, at the end of this section, we give a plan of the rest of the paper.

For a sequence of random variables $(A_n)_{n\geq 0}$ and a sequence $(a_n)_{n\geq 0}$ of real numbers, we write $A_n = O_{\mathbb{P}}(a_n)$ to mean that $(A_n/a_n)_{n\geq 0}$ is tight. We write $A_n = \Theta_{\mathbb{P}}(a_n)$ to mean that $A_n = O_{\mathbb{P}}(a_n)$ and $A_n^{-1} = O_{\mathbb{P}}(a_n^{-1})$. We write $A_n = o_{\mathbb{P}}(a_n)$ to mean $A_n/a_n \xrightarrow{p} 0$ as $n \to \infty$.

2.2.1 The configuration model

We wish to sample a graph uniformly at random from among the graphs with the given degrees $D_1, D_2, \ldots, D_n$. There is a standard method for doing this, which originated (in varying degrees of generality) in the work of Bender and Canfield [28], Bollobás [43] and Wormald [139], called the *configuration model.* (We refer the reader to the recent book of van der Hofstad [135] for a full account of the configuration model and for proofs of the results quoted below.) We begin by first describing the setting where the vertex degrees are deterministic. More precisely, suppose that we have vertices labelled $1, 2, \ldots, n$ where vertex i has degree d_i, for $1 \leq i \leq n$. Suppose that $d_i \geq 1$ for all $1 \leq i \leq n$ and that $\sum_{i=1}^n d_i$ is even. To vertex i, attach d_i *stubs* or *half-edges.* Label the $\sum_{i=1}^n d_i$ half-edges in some arbitrary way, and then choose a pairing of them, uniformly at random. Join the paired half-edges together to make full edges, and then forget the labelling of the half-edges. In general, this procedure yields a multigraph (i.e. with self-loops, or multiple edges). However, if there exist one or more simple graphs with the given degree sequence (that is, if the degree sequence is *graphical*) then, conditionally on the event that the configuration model yields a simple graph, that graph is *uniform* among the possibilities.

We are concerned with the setting where the degrees themselves are independent and identically distributed random variables $D_1, D_2, \ldots, D_n$. An immediate issue is that we cannot guarantee that $\sum_{i=1}^n D_i$ is even. We get around this problem by always assuming that if $\sum_{i=1}^n D_i$ is odd, then we in fact give vertex n degree $D_n + 1$. For the regime and properties in which we are interested, this makes only a negligible difference, and we will ignore it in the sequel. Write $\nu = (\nu_k)_{k\geq 1}$ for the probability mass function of D_1, that is $\nu_k = \mathbb{P}(D_1 = k)$, $k \geq 1$. Let $\mathbf{M}_n(\nu)$ be the multigraph resulting from the configuration model with these degrees. It remains to resolve the issue that the degree sequence may, in principle, be non-graphical. However, it is possible to show that if D_1 has finite variance and $\theta = \theta(\nu) = \mathbb{E}[D_1(D_1 - 1)] / \mathbb{E}[D_1]$ then

$$\lim_{n\to\infty} \mathbb{P}(\mathbf{M}_n(\nu) \text{ is simple}) = \exp(-\theta/2 - \theta^2/4),$$

and the right-hand side is strictly positive (see for instance Proposition 7.13 of [135]). Let $\mathbf{G}_n(\nu)$ be a graph with the distribution of $\mathbf{M}_n(\nu)$ conditioned to be simple; this is our uniform random graph with i.i.d. ν-distributed degrees, and is the main object of study in this paper.

If $\nu_k \sim ck^{-(\alpha+2)}$ for some $\alpha \in (1, 2)$ as $k \to \infty$, we will have that $\max_{1\leq i\leq n} D_i = \Theta_{\mathbb{P}}(n^{1/(\alpha+1)})$. We will see in the sequel that vertices of degree $\Theta(n^{1/(\alpha+1)})$ play an important role in the structure of the graph, and "show up" in the scaling limit as vertices of infinite degree (often known as *hubs*). However, since $\alpha > 1$, with high probability we will not observe edges directly joining two vertices of degree $\Theta(n^{1/(\alpha+1)})$ and, indeed, the vertices of highest degree will be typically well-separated. If $\mathbb{E}\left[D_1^3\right] < \infty$, on the other hand, then $\max_{1\le i\le n} D_i = o_{\mathbb{P}}(n^{1/3})$ and there are no hubs in the limit.

An important property of the configuration model is that the pairing of the edges may be generated in a progressive manner. This makes possible the use of an exploration process in order to capture properties of the (multi-)graph. We do this in a depth-first manner, conditionally on the vertex-degrees, and making use of the arbitrary labelling we gave the half-edges, as follows. Start from a vertex v chosen with probability proportional to its degree D_v. We will maintain a *stack*, namely an ordered list of half-edges which we have seen but not yet explored. Put the D_v half-edges attached to v onto this stack, in increasing order of label, so that the lowest labelled half-edge is on top of the stack. At every subsequent step, if the stack is non-empty, take the top half-edge and sample its pair uniformly at random from those available (i.e. the others on the stack and those which we have not yet observed in our exploration). If the pair half-edge belongs to a vertex w which has not yet been observed (i.e. if the pair half-edge does not lie in the stack), remove the paired half-edges from the system, and add the remaining $D_w - 1$ half-edges attached to w to the top of the stack, again in increasing order of label. If ever the stack becomes empty, select a new vertex with probability proportional to its degree, and put all of its half-edges onto the stack. Repeat until the whole graph has been exhausted. Notice that the stack is empty at the end of a step if and only if we have reached the end of a component, and that in each step except the one at the start of a component, we pair two half-edges. Let $R^n(k)$ be the size of the stack at step k. Then, for example, we may read off the numbers of edges in the successive components as the lengths minus 1 of the excursions above 0 of the process $(R^n(k), k \ge 0)$. It turns out that this process, as we shall explain below, in fact encodes much more information about the multigraph.

Write $|G|$ for the size of the vertex set of a graph G. For a connected graph G, write $s(G)$ for its *surplus*, that is how many more edges it has than any of its spanning trees (which necessarily have $|G| - 1$ edges). Write $G_1^n, G_2^n, \dots$ for the connected components of $\mathbf{G}_n(\nu)$, in decreasing order of size, with ties broken arbitrarily. Similarly, write $M_1^n, M_2^n, \dots$ for the ordered connected components of $\mathbf{M}_n(\nu)$.

2.2.2 The phase transition and critical behaviour of the component sizes

As we have already mentioned, $\mathbf{G}_n(\nu)$ undergoes a phase transition in its component sizes depending on its parameters [112, 113, 90]. Indeed, if $\theta(\nu) \le 1$, then the largest connected component G_1^n of $\mathbf{G}_n(\nu)$ is such that $|G_1^n|/n \xrightarrow{p} 0$. On the other hand, if $\theta(\nu) > 1$ then $|G_1^n|/n \xrightarrow{p} \rho(\nu)$, where $\rho(\nu)$ is some strictly positive constant. These results also hold for $\mathbf{M}_n(\nu)$. To give an idea of why the quantity $\theta(\nu)$ is important, imagine performing the depth-first exploration outlined above, but ignoring any edges which create cycles. Then it is not hard to see that, at each step which is not the start of a component, the degree of the vertex to which the half-edge on the top of the stack connects (as long as it does not connect to something on the stack and thus create a cycle) is a *size-biased pick* from among the remaining possibilities. So, at least close to the beginning, the exploration process should look approximately like a branching process with offspring distribution given by $D^* - 1$, where $\mathbb{P}(D^* = k) = k\nu_k / \mathbb{E}[D_1]$. But then $\theta(\nu) = \mathbb{E}[D^* - 1]$, and so the critical point for the approximating branching process is indeed $\theta(\nu) = 1$. Our interest is in this precisely critical case, and a significant part of this paper is devoted to making the heuristic argument just outlined precise.

The following theorem, due to Joseph [91], summarises some of the possible behaviours for the component sizes in the case $\theta(\nu) = 1$. A version of part (i) was proved independently by Riordan [122] (see below for further discussion). Let

$$\ell^2_{\downarrow} = \left\{ (x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}} : x_1 \geq x_2 \geq \ldots \geq 0, \sum_{i \geq 1} x_i^2 < \infty \right\}.$$

Theorem 2.2.1. *(i) Suppose that* $\mathbb{P}(D_1 = 2) < 1$, $\mathbb{E}[D_1] = \mu$ *and* $\mathbb{E}[D_1(D_1-1)(D_1-2)] = \beta$. *Let B be a standard Brownian motion, and let*

$$\widetilde{L}_t = \sqrt{\frac{\beta}{\mu}} B_t - \frac{\beta}{2\mu^2} t^2, \quad t \geq 0 \qquad \text{and} \qquad R_t = \widetilde{L}_t - \inf_{0 \leq s \leq t} \widetilde{L}_s, \quad t \geq 0. \tag{2.4}$$

Then

$$n^{-2/3}(|G_1^n|, |G_2^n|, \ldots) \xrightarrow{d} (\zeta_1, \zeta_2, \ldots)$$

as $n \to \infty$ in $\ell^2_{\downarrow}$, where $(\zeta_1, \zeta_2, \ldots)$ are the lengths of the excursions above 0 of the process $(R_t)_{t \geq 0}$. The same result holds with $(|G_1^n|, |G_2^n|, \ldots)$ replaced by $(|M_1^n|, |M_2^n|, \ldots)$.

(ii) Suppose that $\lim_{k\to\infty} k^{\alpha+2}\mathbb{P}(D_1 = k) = c$ for some constant $c > 0$ and some $\alpha \in (1, 2)$, and that $\mathbb{E}[D_1] = \mu$. Let X be the process with independent increments whose law is specified by its Laplace transform

$$\mathbb{E}[\exp(-\lambda X_t)] = \exp\left(\int_0^t ds \int_0^\infty dx (e^{-\lambda x} - 1 + \lambda x) \frac{c}{\mu} \frac{1}{x^{\alpha+1}} e^{-xs/\mu} \right), \ \lambda \geq 0, \ t \geq 0,$$

and let

$$\widetilde{L}_t = X_t - \frac{c\Gamma(2-\alpha)}{\alpha(\alpha-1)\mu^\alpha} t^\alpha, \quad t \geq 0 \qquad \text{and} \qquad R_t = \widetilde{L}_t - \inf_{0 \leq s \leq t} \widetilde{L}_s \quad t \geq 0. \tag{2.5}$$

Then

$$n^{-\alpha/(\alpha+1)}(|M_1^n|, |M_2^n|, \ldots) \xrightarrow{d} (\zeta_1, \zeta_2, \ldots)$$

as $n \to \infty$ in ℓ^2, where $(\zeta_1, \zeta_2, \ldots)$ are the lengths of the excursions above 0 of the process $(R_t)_{t \geq 0}$.

The sequences $(\zeta_1, \zeta_2, \ldots)$ appearing in Theorem 2.2.1 must, of course, have the same distributions as the lengths of the excursions above 0 of the processes $(R_t, t \geq 0)$ from the Introduction. Indeed, the processes $\tilde{L}$ defined in (2.4) and (2.5) have the same distributions as those defined at (2.2) and (2.1), respectively. We prove this in Proposition 2.3.2 below.

Joseph [91] conjectures that Theorem 2.2.1 (ii) should also hold with $\mathbf{M}_n(\nu)$ replaced by $\mathbf{G}_n(\nu)$. We show in the sequel (Section 2.5.3) that this is indeed true (this has been proved independently by Dhara, van der Hofstad, van Leeuwaarden and Sen [62]). In consequence, all of our scaling limit results hold interchangeably for $\mathbf{G}_n(\nu)$ and $\mathbf{M}_n(\nu)$.

The common structure exhibited by the two parts of Theorem 2.2.1 is no coincidence. In both cases, the proof proceeds via an exploration of the graph similar to the one described earlier. As outlined above, locally, the components resemble critical branching processes. Since the components have small surplus, the excursions of the stack-size process above 0 approximately encode the component sizes. Moreover, the stack-size process behaves approximately like a reflected random walk. A weak convergence result for the stack-size process then yields the convergence of the component sizes.

Riordan [122], in fact, proves a more refined version of Theorem 2.2.1 (i), but under the (non-optimal) assumption that the degrees are bounded. Firstly, his results are stated for a uniform random graph with a given n-dependent deterministic degree sequence $(d_i^{(n)})_{i \geq 1}$, where the moment conditions on D_1 are replaced by appropriate convergence results for the moments of the degree of a uniformly chosen vertex. In particular, he is able to consider the components anywhere in the critical window, rather than precisely at $\theta = 1$. (We refer the reader to [122] for the details.) Secondly, he takes account also of the surplus of each component. Jointly with the convergence of the rescaled component sizes, he shows that

$$(s(G_1^n), s(G_2^n), \ldots) \xrightarrow{d} (M_1, M_2, \ldots)$$

for a non-trivial random sequence $(M_1, M_2, \ldots) \in \mathbb{Z}_+^{\mathbb{N}}$. The sequence $(M_1, M_2, \ldots)$ is again obtained using the process R in (2.4): on top of the graph of the random function R, superpose a Poisson point process of intensity $1/\mu$ in the plane. Then M_i is the number of points falling in the area beneath the excursion $\widetilde{\varepsilon}_i$ and above the x-axis, for $i \geq 1$.

The first result of this kind was proved by Aldous [13] for the Erdős-Rényi random graph, $G(n, p)$ at its critical point. More precisely, consider the graph $\mathbf{G}_n^{\mathrm{ER}}$ obtained by taking n vertices and connecting any pair of them by an edge independently with probability $p = 1/n$. Write $G_1^{\mathrm{ER},n}, G_2^{\mathrm{ER},n}, \ldots$ for the components listed in decreasing order of size. Define $\widetilde{L}$ and R as in (2.4) with $\beta = \mu = 1$, let $\zeta_1, \zeta_2, \ldots$ be the lengths of the excursions of R and let $M_1, M_2, \ldots$ be the numbers of points of a Poisson process of intensity 1 falling in each excursion.

Theorem 2.2.2 (Aldous [13]). *As $n \to \infty$,*

$$\left(n^{-2/3}(|G_1^{\mathrm{ER},n}|, |G_2^{\mathrm{ER},n}|, \ldots), (s(G_1^{\mathrm{ER},n}), s(G_1^{\mathrm{ER},n}), \ldots)\right) \xrightarrow{d} ((\zeta_1, \zeta_2, \ldots), (M_1, M_2, \ldots))$$

where the convergence is in ℓ^2 for the component sizes and in the sense of the product topology for the surpluses.

The limit is the same as in Theorem 2.2.1 (i) in the case $\beta = \mu = 1$. This should be intuitively unsurprising, since the vertex degrees in the Erdős–Rényi random graph approximately behave like i.i.d. Poisson(1) random variables, for which Theorem 2.2.1 would apply with $\beta = \mu = 1$. (Aldous' theorem in fact treats the whole critical window, i.e. $G(n, 1/n + \lambda n^{-4/3})$ for $\lambda \in \mathbb{R}$. The effect is to introduce an extra drift of λt into the process $\widetilde{L}$; we omit the very similar details for the sake of brevity.)

2.2.3 Branching processes and their metric space scaling limits

As alluded to above, the components of our critical random graphs behave approximately like critical branching process trees. It will be useful to spend a little time now exploring what happens in the true branching process setting, since what we do later will be analogous. Suppose that we take a sequence of i.i.d. Galton–Watson trees, with offspring distribution represented by some non-negative random variable Y with $\mathbb{E}[Y] = 1$ and $\mathbb{P}(Y = 1) < 1$. (This entails that each of the trees has finite size almost surely.) We use the standard encoding of this forest in terms of its *Lukasiewicz path* or *depth-first walk*, given by $S(0) = 0$ and $S(k) = \sum_{i=1}^{k}(Y_i - 1)$ for $k \geq 1$ (see Le Gall [95] or Duquesne and Le Gall [70] for more details). Here, as usual, we explore the vertices of the forest in depth-first order, and Y_i is the number of children of the ith vertex that we visit; these get added to the stack to await processing. The stack-size process is essentially a reflected version of S, given by $(1 + S(k) - \min_{0 \leq j \leq k} S(j))_{k \geq 0}$. It is straightforward to see that the individual trees correspond to excursions above the running minimum of $(S(k))_{k \geq 0}$; it is technically easier to work with the depth-first walk than with the stack-size process, since S it is an *unreflected* random walk. An even more convenient encoding of the forest is given by the height process, which tracks the generation of the successive vertices listed in depth-first order. (It is, however, considerably harder to understand its distribution.) In terms of the depth-first walk, the height process $(G(n))_{n \geq 0}$ is defined by $G(0) = 0$ and

$$G(n) := \# \left\{ j \in \{0, 1, \ldots, n-1\} : S(j) = \min_{j \leq k \leq n} S(k) \right\}. \tag{2.6}$$

The different trees now correspond to excursions above 0 of G.

The following generalised functional central limit theorem indicates some of the possible scaling limits for S in this setting (see, for example, Theorem 3.7.2 of Durrett [71]).

Theorem 2.2.3. *(i) Suppose that $\mathbb{E}[Y] = \sigma^2 < \infty$. Then*

$$n^{-1/2}(S(\lfloor nt \rfloor), t \geq 0) \xrightarrow{d} \sigma(B_t, t \geq 0)$$

as $n \to \infty$, in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, where B is a standard Brownian motion.

(ii) Suppose that $\lim_{k\to\infty} k^{\alpha+1}\mathbb{P}(Y=k) = c$ *for some constant* $c > 0$ *and some* $\alpha \in (1,2)$. *Then*

$$n^{-1/\alpha}(S(\lfloor nt \rfloor), t \geq 0) \xrightarrow{d} (L_t, t \geq 0)$$

as $n \to \infty$, *in* $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, *where* L *is a spectrally positive* α*-stable Lévy process, with Laplace transform*

$$\mathbb{E}[\exp(-\lambda L_t)] = \exp\left(\frac{c\Gamma(2-\alpha)}{\alpha(\alpha-1)}\lambda^\alpha t\right), \quad \lambda \geq 0, \quad t \geq 0.$$

We now turn to the behaviour of the height process. In the Brownian case, this turns out to be asymptotically the same as that of the reflected depth-first walk, up to a scaling constant. In the stable case, however, matters are a little more complicated. Consider the α-stable Lévy process L. Chapter 1 of Duquesne & Le Gall [70] shows that it is possible to make sense of a corresponding continuous height process, defined as follows. First, for $0 \leq s \leq t$, let $\hat{L}_s^{(t)} = L_t - L_{(t-s)-}$ and let $\hat{M}_s^{(t)} = \sup_{0 \leq r \leq s} \hat{L}_r^{(t)}$. Then define H_t to be the local time at level 0 of the process $\hat{L}^{(t)} - \hat{M}^{(t)}$. We may choose the normalization in such a way that

$$H_t = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^t \mathbb{1}_{\{\hat{M}_s^{(t)} - \hat{L}_s^{(t)} \leq \epsilon\}} ds \tag{2.7}$$

in probability. Theorem 1.4.3 of [70] shows that H has continuous sample paths with probability 1, and so we may (and will) work with a continuous version in the sequel. Corollary 2.5.1 of [70] entails the following joint convergences.

Theorem 2.2.4 (Duquesne & Le Gall [70]). *(i) Suppose that* $\mathbb{E}[Y] = \sigma^2 < \infty$. *Then*

$$(n^{-1/2}S(\lfloor nt \rfloor), n^{-1/2}G(\lfloor nt \rfloor), t \geq 0) \xrightarrow{d} \left(\sigma B_t, \frac{2}{\sigma}\left(B_t - \inf_{0 \leq s \leq t} B_s\right), t \geq 0\right).$$

as $n \to \infty$ *in* $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^2)$.

(ii) Suppose that $\lim_{k\to\infty} k^{\alpha+1}\mathbb{P}(Y=k) = c$ *for some constant* $c > 0$ *and some* $\alpha \in (1,2)$. *Then we have*

$$\left(n^{-1/\alpha}S(\lfloor nt \rfloor), n^{-(\alpha-1)/\alpha}G(\lfloor nt \rfloor), t \geq 0\right) \xrightarrow{d} (L_t, H_t, t \geq 0)$$

as $n \to \infty$ *in* $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^2)$.

There is also a conditional version of Theorem 2.2.4 for the depth-first walk S^n and height process G^n of a *single* Galton–Watson tree, conditioned to have total progeny n. (Let us assume that $\mathbb{P}(Y=k) > 0$ for all $k \geq 0$, so that the event of having total progeny n has positive probability for all n; this is not really necessary, but will facilitate the statement of the theorem.)

Theorem 2.2.5. *(i)* (Marckert & Mokkadem [107]). *Suppose that* $\mathbb{E}[Y] = \sigma^2 < \infty$. *Then*

$$(n^{-1/2}S^n(\lfloor nt \rfloor), n^{-1/2}G^n(\lfloor nt \rfloor), 0 \leq t \leq 1) \xrightarrow{d} \left(\sigma \mathbb{e}(t), \frac{2}{\sigma}\mathbb{e}(t), 0 \leq t \leq 1\right),$$

as $n \to \infty$ *in* $\mathbb{D}([0,1], \mathbb{R}^2)$, *where* $\mathbb{e}$ *is a standard Brownian excursion.*

(ii) (Duquesne [69]). *Suppose that* $\lim_{k\to\infty} k^{\alpha+1}\mathbb{P}(Y=k) = c$ *for some constant* $c > 0$ *and some* $\alpha \in (1,2)$. *Then we have*

$$\left(n^{-1/\alpha} S^n(\lfloor nt \rfloor), n^{-(\alpha-1)/\alpha} G^n(\lfloor nt \rfloor), t \geq 0\right) \xrightarrow{d} (\mathbb{e}(t), \mathbb{h}(t), 0 \leq t \leq 1)$$

as $n \to \infty$ *in* $\mathbb{D}([0,1], \mathbb{R}^2)$, *where* $\mathbb{e}$ *is a normalised excursion of* L *and* $\mathbb{h}$ *is the corresponding normalised excursion of* H.

We now describe briefly how a limiting height process excursion may be used to define a limit $\mathbb{R}$-tree (and the reader to the survey paper of Le Gall [95] for more details). Suppose first that $h : [0, \zeta_h] \to \mathbb{R}_+$ is any continuous function such that $h(0) = h(\zeta_h) = 0$. Define a pseudo-metric on $[0,1]$ via

$$d_h(x,y) = h(x) + h(y) - 2 \min_{x \wedge y \leq z \leq x \vee y} h(z), \quad x, y \in [0, \zeta_h].$$

Define an equivalence relation $\sim$ on $[0, \zeta_h]$ by declaring $x \sim y$ if $d_h(x,y) = 0$. Now let $\mathcal{T}_h = [0, \zeta_h]/\sim$ and endow it with the distance d_h in order to obtain a metric space (which is compact: one checks easily that it is a Hausdorff space, and that it is sequentially compact). Then $(\mathcal{T}_h, d_h)$ is the $\mathbb{R}$-tree encoded by h. Write $p_h : [0, \zeta_h] \to \mathcal{T}_h$ for the canonical projection. We may additionally endow $(\mathcal{T}_h, d_h)$ with a natural "uniform" measure μ_h having total mass ζ_h, obtained as the push-forward of the Lebesgue measure on $[0, \zeta_h]$ onto the tree (and concentrated on the leaves of $\mathcal{T}_h$). Write $\mathbb{M}$ for the space of compact metric spaces each endowed with a finite (non-negative) Borel measure, up to measure-preserving isometry. We equip $\mathbb{M}$ with the Gromov–Hausdorff–Prokhorov distance $\mathrm{d}_{\mathrm{GHP}}$, defined as follows. (See Section 2.1 of [7] for more details and proofs of the results claimed below, as well as further references to the literature.) Let (X, d, μ) and (X', d', μ') be elements of $\mathbb{M}$. We say that C is a *correspondence* between X and X' if $C \subseteq X \times X'$ and, whenever $x \in X$, there exists $x' \in X'$ such that $(x, x') \in C$ and vice versa. The *distortion* of the correspondence C is

$$\mathrm{dist}(C) := \sup\{|d(x_1, x_2) - d'(x_1', x_2')| : (x_1, x_1'), (x_2, x_2') \in C\}.$$

Write $\mathcal{C}(X, X')$ for the set of correspondences between X and X'. Write $M(X, X')$ for the set of non-negative Borel measures on $X \times X'$. Write p and p' for the canonical projections from $X \times X'$ to X and X' respectively. We define the *discrepancy* of $\pi \in M(X, X')$ with respect to μ and μ' to be

$$\mathrm{disc}(\pi; \mu, \mu') = \|\mu - p_* \pi\| + \|\mu' - p'_* \pi\|,$$

where $\|\nu\|$ is the total variation of the signed measure ν. We define the Gromov–Hausdorff–Prokhorov distance by

$$\mathrm{d}_{\mathrm{GHP}}((X, d, \mu), (X', d', \mu')) := \inf_{C \in \mathcal{C}(X, X'),\ \pi \in M(X, X')} \left\{ \frac{1}{2} \mathrm{dist}(C) \vee \mathrm{disc}(\pi; \mu, \mu') \vee \pi(C^c) \right\}.$$

Then $(\mathbb{M}, \mathrm{d_{GHP}})$ is a Polish space. We observe a very useful upper bound for the Gromov–Hausdorff–Prokhorov distance between $\mathbb{R}$-trees encoded by continuous excursions:

$$\begin{aligned} &\mathrm{d_{GHP}}((\mathcal{T}_h, d_h, \mu_h), (\mathcal{T}_g, d_g, \mu_g)) \\ &\quad \leq 2 \max\left\{ \sup_{0 \leq x \leq \zeta_h \wedge \zeta_g} |h(x) - g(x)|, \sup_{\zeta_h \wedge \zeta_g < x \leq \zeta_h} h(x) + \sup_{\zeta_h \wedge \zeta_g < x \leq \zeta_g} g(x), \frac{1}{2}|\zeta_h - \zeta_g| \right\}, \end{aligned} \tag{2.8}$$

The *random* $\mathbb{R}$-trees encoded by $2\mathbb{e}$ for $\alpha = 2$ and $\mathbb{h}$ for $\alpha \in (1, 2)$ are known as the *Brownian continuum random tree*, for which we will write $(\mathcal{T}^{(2)}, d^{(2)})$ (with mass measure $\mu^{(2)}$), and the *α-stable tree*, for which we will write $(\mathcal{T}^{(\alpha)}, d^{(\alpha)})$ (with mass measure $\mu^{(\alpha)}$), respectively. (Note that because of our choice of Laplace exponent, this is a constant multiple of the usual α-stable tree.)

Let T_n be our Galton–Watson tree conditioned to have size n. The natural way to take a scaling limit of the tree itself is to consider it as a metric space using the graph distance d_n. Create a (probability) measure μ_n by assigning mass $1/n$ to each vertex of T_n. An important consequence of Theorem 2.2.5 and the bound (2.8) is the following.

Theorem 2.2.6. *(i) Suppose that* $\mathbb{E}[Y] = \sigma^2 < \infty$. *Then*

$$\left(T_n, \frac{\sigma}{\sqrt{n}} d_n, \mu_n\right) \xrightarrow{d} \left(\mathcal{T}^{(2)}, d^{(2)}, \mu^{(2)}\right).$$

as $n \to \infty$ *for the Gromov–Hausdorff–Prokhorov topology.*

(ii) Suppose that $\lim_{k\to\infty} k^{\alpha+1}\mathbb{P}(Y = k) = c$ *for some constant* $c > 0$ *and some* $\alpha \in (1, 2)$. *Then*

$$\left(T_n, n^{-(\alpha-1)/\alpha} d_n, \mu_n\right) \xrightarrow{d} \left(\mathcal{T}^{(\alpha)}, d^{(\alpha)}, \mu^{(\alpha)}\right),$$

as $n \to \infty$ *for the Gromov–Hausdorff–Prokhorov topology.*

Returning now to the setting of Theorem 2.2.4, the excursions of the limiting height process can heuristically be thought of as defining a forest of random $\mathbb{R}$-trees. (Since there is neither a shortest nor a longest excursion, there is no sensible way to list these trees. For definiteness, let us instead think of restricting to an interval $[0, t]$ in time, for which there *is* a longest excursion, and then list the trees in decreasing order of size.) Using the scaling properties of the underlying Lévy processes, these consist of randomly rescaled copies of the Brownian continuum random tree in case (i) or α-stable trees in case (ii), respectively. We refer to these as the *Brownian* and *stable forests.*

2.2.4 Our method

Our approach to proving Theorem 2.1.1 is as follows. Firstly, we show that the law of the depth-first walk of the graph is (up to a small error) absolutely continuous with respect to that of a centred random walk which is in the domain of attraction of the spectrally positive α-stable

Lévy process L. This enables us to give an alternative (and perhaps more "conceptual") proof of Theorem 2.2.1. We also show that the convergence of the depth-first walk can be boosted to a joint convergence with the corresponding height process. The joint convergence of this pair of coding functions in the setting of a sequence of i.i.d. Galton–Watson trees, Theorem 2.2.4, is a highly non-trivial result. The corresponding result in our setting, however, follows relatively straightforwardly from Theorem 2.2.4 via absolute continuity and some integrability lemmas.

The height process is the key ingredient in proving a metric space convergence for these graphs, and allows us to show the convergence of a spanning forest of our graph. In order to obtain the full metric space convergence, we must also control the edges which form cycles. We call these *back-edges*. We prove that the number of back-edges edges in the "large components" is a tight quantity. This firstly allows us to resolve Conjecture 8.5 of Joseph [91], by showing that all of the above results extend to the case where the multigraph is conditioned to be simple. Secondly, we are able to capture the full graph structure by tracking also the locations of these back-edges in the spanning forest. We finally show that all of these quantities can be passed through to the limit in such a way that we get convergence to the stable graph.

2.2.5 Related work on scaling limits of critical random graphs, universality, and open problems

This paper is a contribution to a now extensive literature on scaling limits of critical random graphs. In this section, we will place our work in context by giving a summary of related results.

As mentioned above, the first critical random graph to be studied from the perspective of scaling limits was the Erdős-Rényi random graph, in the work of Aldous [13], who considered both component sizes and surpluses. Addario-Berry, Broutin and Goldschmidt [6, 5] built on Aldous' work in order to prove convergence to the $\beta = \mu = 1$ Brownian graph, in the sense of an ℓ^4 version of the Gromov–Hausdorff distance. (It is straightforward to improve this to a convergence in an ℓ^4 version of the Gromov–Hausdorff–Prokhorov distance, which appears as Theorem 4.1 of Addario-Berry, Broutin, Goldschmidt & Miermont [7].)

Several models have been proved to lie in the same universality class as the Erdős–Rényi random graph, which is roughly characterised by the property that the degree of a uniformly chosen vertex converges to a limit with finite third moment. Already in [13], Aldous had, in fact, also considered another model: a rank-one inhomogeneous random graph in which, for each $n \geq 1$, we are given a sequence of weights $\mathbf{w}^{(n)} = (w_1^{(n)}, w_2^{(n)}, \ldots, w_n^{(n)})$ and each pair of vertices $\{i, j\}$ is connected independently with probability $1 - \exp(-q^{(n)} w_i^{(n)} w_j^{(n)})$, for $1 \leq i, j \leq n$. Such graphs may be constructed dynamically by assigning an exponential clock to each potential edge and including the edge when the clock rings. It is straightforward to see that, in consequence, the component sizes then evolve according to the multiplicative coalescent. In his Proposition 4, Aldous gave conditions on sequences $(\mathbf{w}^{(n)}, q^{(n)})_{n \geq 1}$ for which one gets convergence of the rescaled component *weights* to the same limit as for the component sizes in the Erdős–Rényi case. These results were generalised by Bhamidi, van der Hofstad and van Leeuwaarden [40] to give convergence of the rescaled component *sizes* in the Norros–Reittu model [119] (for which $q^{(n)}$ above is replaced by $1/\sum_{i=1}^{n} w_i^{(n)}$) to the sequence $(\zeta_1, \zeta_2, \ldots)$ appearing in Theorem 2.2.1 (i), with a general β and μ. The convergence of the component sizes was also treated in a similar setting but with i.i.d. vertex weights by Turova [134].

Nachmias and Peres [115] proved the convergence of the rescaled component sizes for critical percolation on a random d-regular graph, for $d \geq 3$, to the excursion lengths of the reflected Brownian motion with parabolic drift for appropriate β and μ. As we have already detailed above, Riordan [122] and Joseph [91] proved analogous results for the critical configuration model with asymptotic degree distribution possessing finite third moment, with Riordan treating the surpluses as well as the component sizes. Dhara, van der Hofstad, van Leeuwaarden and Sen [61] improved these results to give the scaling limit of the sizes and the surpluses under a minimal set of conditions on the (deterministic) vertex degrees, which essentially amount to the convergence in distribution of the degree of a uniform vertex, along with the convergence of its third moment. In a somewhat different direction, Bhamidi, Budhiraja and Wang [33] considered critical random graphs generated by Achlioptas processes [42] with bounded size rules. They again proved convergence of the rescaled component sizes, along with the surpluses, as a process evolving in the critical window, building on results for the barely subcritical regime proved in [34]. Federico [76] has recently proved a scaling limit for the component sizes of the *random intersection graph* which is related to that of the Erdős–Rényi model.

Turning now to the metric structure, very general results concerning the domain of attraction of the Brownian graph have been proved by Bhamidi, Broutin, Sen and Wang [32], building on earlier work for the Norros–Reittu model by Bhamidi, Sen and Wang [38]. In particular, [32] gives a set of sufficient conditions under which one obtains convergence in the Gromov–Hausdorff–Prokhorov sense to the Brownian graph. It is also demonstrated in that paper that these conditions are fulfilled for certain critical inhomogeneous random graphs (of the stochastic block model variety), and for critical percolation on a supercritical configuration model with finite third moment degree distribution. A crucial role is played by dynamical constructions of the graphs in question, and by the idea that some pertinent statistic of the evolving graph may be well approximated by the multiplicative coalescent. Bhamidi and Sen [37] later proved convergence to the Brownian graph for the critical configuration model (rather than for percolation on the supercritical case) in the Gromov–Hausdorff–Prokhorov sense, under the same set of minimal conditions as in [61], and used it to deduce geometric properties of the *vacant set* left by a random walk on various models of graph.

We have mentioned a few examples of critical percolation on graphs for which the resulting cluster sizes lie in the universality class of the Brownian graph. This is expected to be true in much greater generality: for a wide variety of finite base graphs which are sufficiently "high dimensional", although the percolation critical point will be model-dependent, the behaviour in the vicinity of that critical point should essentially be the same as in the mean-field case of percolation on the complete graph, i.e. the Erdős–Rényi model. We refer the reader to the book of Heydenreich and van der Hofstad [86] for an in-depth discussion of this universality conjecture.

The results of the present paper primarily concern cases where the degree of a uniformly chosen vertex has infinite third moment and a power-law tail with exponent $\alpha + 1 \in (2, 3)$, and in this context the picture is more complicated. As in the Brownian case, it is to be expected that, as long as the degree of a uniformly chosen vertex has the right properties, we should get the same scaling limit irrespective of precisely which model we consider. It is technically more straightforward to consider rank-one inhomogeneous random graphs than the configuration model. In the context of component sizes, this was first done by Aldous and Limic [15] for the rank-one model treated by Aldous in [13] but with appropriately altered conditions on the weight sequence. These different conditions correspond to different extremal entrance laws for the multiplicative coalescent. Aldous and Limic obtained the analogue of Theorem 2.2.1 for the component *weights*, where the limit is now given by the ordered lengths of the excursions above the running infimum of the *thinned Lévy process*,

$$\left(\kappa B_t + \lambda t + \sum_{i \geq 1} \vartheta_i (\mathbb{1}_{\{E_i \leq t\}} - \vartheta_i t)\right)_{t \geq 0}, \tag{2.9}$$

where $E_i \sim \text{Exp}(\vartheta_i)$ for each $i \geq 1$, $\kappa \geq 0$, $\lambda \in \mathbb{R}$, and $\vartheta_1 \geq \vartheta_2 \geq \ldots \geq 0$ is a sequence such that $\sum_{i \geq 1} \vartheta_i^3 < \infty$ and, if $\kappa = 0$, also $\sum_{i \geq 1} \vartheta_i^2 = \infty$. This was extended in the $\kappa = 0$ case by Bhamidi, van der Hofstad and van Leeuwaarden [41] to give the convergence of the component *sizes* for the Norros–Reittu model with a specific weight sequence. Heuristically, the choice of entrance law for the multiplicative coalescent is determined by the properties of the barely subcritical graph. For the configuration model, the first work in the power-law setting was that of Joseph [91] detailed above for the case of i.i.d. degrees. The convergence of the sizes and surpluses for much more general (deterministic) degree sequences were treated by Dhara, van der Hofstad, van Leeuwaarden and Sen [62], with the possible scaling limits being driven by the same $\kappa = 0$ thinned Lévy processes as in the Norros–Reittu model.

A significant challenge in obtaining a metric space convergence in the power-law setting is that one often does not have direct access to a scaling limit result for the height process of the spanning forest discovered by a depth-first exploration. (That we have such a result in the case of i.i.d. degrees is of considerable help to us.) The first metric space scaling limit in the power-law setting was obtained by Bhamidi, van der Hofstad and Sen [39] for the Norros–Reittu model with the specific weight sequence used in [41]. Here, the convergence is in the product Gromov–Hausdorff–Prokhorov sense, and the limit object is constructed by making vertex identifications in tilted inhomogeneous continuum random trees (of the sort introduced by Aldous and Pitman in [16]).

Broutin, Duquesne and Wang [50, 49] use a very different approach in order to prove a unified metric space scaling limit for the Norros–Reittu model with very general weight sequences. They are able to treat situations where the scaling limit of the depth-first walk is a thinned Lévy process for any $\kappa \geq 0$, $\lambda \in \mathbb{R}$ and sequence $(\vartheta_1, \vartheta_2, \ldots)$, recovering the generality of Aldous and Limic's paper [15]. They embed spanning subtrees of the components of the graph inside a forest of Galton–Watson trees, and exploit the convergence of this (bigger) forest on rescaling to the sequence of $\mathbb{R}$-trees encoded by a Lévy process (as in Duquesne and Le Gall [70]), whose height process also converges. This enables them to obtain the convergence of the height process of the true spanning forest in the Gromov–Hausdorff–Prokhorov sense; the surplus edges can also be tracked, in order to obtain a product Gromov–Hausdorff–Prokhorov convergence of the whole ordered sequence of graph components.

Let us finally turn to the work of Bhamidi, Dhara, van der Hofstad and Sen [36], who proved a metric space scaling limit analogous to that of [39] for critical percolation on a supercritical configuration model. Among the settings studied so far, theirs is the closest to ours, although the technical content is very different (their work relies on the analysis of some susceptibility functions in the barely subcritical regime, while we do not study the latter). We will describe it precisely, in order to provide a comparison with Theorem 2.1.1. They take a (deterministic) degree sequence $\mathfrak{d}_1^n, \mathfrak{d}_2^n, \ldots, \mathfrak{d}_n^n$ such that $\sum_{i=1}^n \mathfrak{d}_i^n$ is even and, if $\mathfrak{D}_n$ is the degree of a typical vertex, then

(i) $n^{-1/(\alpha+1)} \mathfrak{d}_i^n \to \vartheta_i$ as $n \to \infty$ for each $i \geq 1$, where $\vartheta_1 \geq \vartheta_2 \geq \ldots \geq 0$ is such that $\sum_{i\geq 1} \vartheta_i^3 < \infty$ but $\sum_{i \geq 1} \vartheta_i^2 = \infty$;

(ii) $\mathfrak{D}_n \xrightarrow{d} \mathfrak{D}$ as $n \to \infty$, along with the convergence of its first two moments, for some random variable $\mathfrak{D}$ with $\mathbb{P}(\mathfrak{D} = 1) > 0$, $\mathbb{E}[\mathfrak{D}] = \mu$ and $\mathbb{E}[\mathfrak{D}(\mathfrak{D}-1)]/\mathbb{E}[\mathfrak{D}] = \theta > 1$, and
$$\lim_{K\to\infty} \limsup_{n\to\infty} n^{-3/(\alpha+1)} \sum_{i \geq K+1} (\mathfrak{d}_i^n)^3 = 0.$$

Let $\theta_n = \mathbb{E}[\mathfrak{D}_n(\mathfrak{D}_n - 1)]/\mathbb{E}[\mathfrak{D}_n]$ (which, by (ii), converges to $\theta > 1$). They then perform percolation at parameter
$$p_n(\lambda) = \frac{1}{\theta_n} + \lambda n^{-(\alpha-1)/(\alpha+1)},$$
for some $\lambda \in \mathbb{R}$, which yields a graph in the critical window. In this setting, their Theorem 2.2 is the precise analogue of our Theorem 2.1.1 but with the convergence in the product Gromov-weak topology and with the limit object $((\mathcal{G}_i, d_i, \mu_i), i \geq 1)$ constructed by making vertex identifications in the tilted inhomogeneous continuum random trees mentioned above. (We understand that this convergence will be improved to a product Gromov–Hausdorff–Prokhorov convergence under an extra technical condition in work in preparation [35].) A precise description of the limit object would be too lengthy to undertake here, but it is instructive to compare the scaling limit of the depth-first walk in the two settings. For us, this is the measure-changed stable Lévy

process $\widetilde{L}$; for Bhamidi, Dhara, van der Hofstad and Sen it is the thinned Lévy process in (2.9) with $\kappa = 0$. To make the connection between the results, suppose now we take $\mathfrak{D}$ such that $\mathbb{P}(\mathfrak{D} = 1) > 0$, $\mathbb{E}[\mathfrak{D}] = \mu$, $\mathbb{E}[\mathfrak{D}(\mathfrak{D}-1)]/\mathbb{E}[\mathfrak{D}] = \theta > 1$ and $\mathbb{P}(\mathfrak{D} = k) \sim ck^{-\alpha-2}$. Let $\mathfrak{d}_1^n, \ldots, \mathfrak{d}_n^n$ be an ordered sample of i.i.d. random variables $\mathfrak{D}_1, \ldots, \mathfrak{D}_n$ with the same distribution as $\mathfrak{D}$. Then conditions (i) and (ii) above are satisfied almost surely for some sequence of random variables $\vartheta_1, \vartheta_2, \ldots$ (see Section 2.2 of [62]). Perform percolation at parameter $p = 1/\theta$ to obtain new degrees $D_1, D_2, \ldots, D_n$ which are mildly dependent but whose ordered version behaves very similarly to the order statistics of a i.i.d. sample which satisfy the conditions of our theorem. (In particular, using results of Janson [89], such mild dependence can be shown to have a negligible effect on the properties of the graph.) Then it should be the case that Bhamidi, Dhara, van der Hofstad and Sen's limit object is the same as the stable graph. In particular, the process defined at (2.9) with $\kappa = 0$ and $\lambda = 0$ should, for this particular random sequence $(\vartheta_1, \vartheta_2, \ldots)$, have the same law as $\widetilde{L}$. Similarly, if it is the case that the scaling limit is the same as for the analogous inhomogeneous random graph setting, then our limit object should also coincide with a particular annealed version of that of Broutin, Duquesne and Wang [50, 49].

It is perhaps worth emphasising that, in contrast to the bulk of the other papers cited here, the multiplicative coalescent (and its relationship to percolation) appears nowhere in our proofs, and is conceptually absent from our approach.

Let us now give a list of open problems and conjectures arising from our work.

(i) Prove that the stable graph is, indeed, an annealed version of the limit object from [36] or [50, 49].

(ii) The convergence in our main theorem occurs with respect to the product Gromov–Hausdorff–Prokhorov topology. For sequences $\mathbf{A} = (A_1, A_2, \ldots)$ and $\mathbf{B} = (B_1, B_2, \ldots)$ of compact measured metric spaces, we may obtain stronger topologies using the distances

$$\mathrm{dist}_p(\mathbf{A}, \mathbf{B}) = \left(\sum_{i \geq 1} d_{\mathrm{GHP}}(A_i, B_i)^p \right)^{1/p} \tag{2.10}$$

for $p \geq 1$. For the Erdős–Rényi random graph, the analogous convergence to the Brownian graph holds in the sense of dist_4. We conjecture that it should be possible to improve our main result for $\alpha \in (1, 2]$ to a convergence in the sense of $\mathrm{dist}_{2\alpha/(\alpha-1)}$.

(iii) One reason for wanting to prove such a result is that it would imply the convergence in distribution of the diameter of the whole graph (i.e. the largest distance between any two vertices in the same component). In order to prove convergence in $\mathrm{dist}_{2\alpha/(\alpha-1)}$, we would need bounds on the component diameters in terms of powers of their sizes for the whole graph (we can do this for the parts explored up to time $O(n^{\alpha/(\alpha+1)})$ using the methods of this paper, but that is not sufficient). A finer understanding of the *barely subcritical regime* for the configuration model would presumably help to resolve this issue.

(iv) As shown in Proposition 2.6.2, the measure change used in this paper makes sense for a large family of spectrally positive Lévy processes (see Section 2.6.1 for the precise conditions). Any such Lévy process may be intuitively thought of as encoding a forest of continuum trees, although the analogue of Theorem 2.2.4 holds only with the imposition of extra regularity conditions (see Theorem 2.3.1 of [70]). Is it possible to find a sequence of degree distributions $(\nu^n)_{n\geq 1}$, depending now on n and such that the regularity conditions are satisfied, so that if we take $D_1^{(n)}, \ldots, D_n^{(n)}$ to be i.i.d. random variables with distribution ν^n then we get convergence of our discrete measure change to its continuum analogue? Or does the self-similarity inherent in the Brownian and stable settings play a key role? If a generalisation to the Lévy case is possible, what is the connection to thinned Lévy processes, or to the approach of Broutin, Duquesne and Wang [50, 49]?

For simplicity we have restricted our attention in this paper to the case where $\theta(\nu) = 1$. The critical window is obtained by considering the situation where the degrees $D_1^n, \ldots, D_n^n$ are i.i.d. but now with some n-dependent degree distribution ν^n, such that $\mathbb{E}[D_1^n] \to \mu$ for some μ as $n \to \infty$, $\theta(\nu^n) = 1 + \lambda n^{-(\alpha-1)/(\alpha+1)}$ and $\mathbb{P}(D_1^n = k) \sim ck^{-\alpha-2}$ as $k \to \infty$, for some fixed $\lambda \in \mathbb{R}$. This regime is the subject of work in progress by Serte Donderwinkel.

2.2.6 Plan of the rest of the paper

In Section 2.3, we study the process $\widetilde{L}$ which gives rise to the stable graph. In particular, we establish the local absolute continuity relation between $\widetilde{L}$ and L, and present some results in excursion theory. The section concludes with the proof of Theorem 2.1.2. In Section 2.4, we study a forest which is closely related to $\mathbf{M}_n(\nu)$. We show that the absolute continuity relation (2.1), (2.2) may be seen as the limit of a discrete measure change between the degrees in the order we observe them when we explore this forest in a depth-first manner and an i.i.d. sequence of random variables whose law is the size-biased version of ν. The main result of this section is Theorem 2.4.1, which gives the joint convergence of the depth-first walk and height process of the discrete forest to their continuum counterparts. In Section 2.5, we explore the multigraph $\mathbf{M}_n(\nu)$ in a depth-first manner, and record its structure via coding functions close to those of the forest in Section 2.4, and show their convergence in law. We also deal with the occurrence of the back-edges, and prove that $\mathbf{M}_n(\nu)$ and $\mathbf{G}_n(\nu)$ cannot have different scaling limits. We must then extract the individual components of the graph in decreasing order of size, and prove that their individual coding functions converge. We adapt an approach of Aldous [13] using size-biased point processes; this is perhaps the most technical part of the paper. Section 2.5 culminates in the proof of Theorem 2.1.1. The Appendix contains various technical results. In particular, in Section 2.6.1 we give a formulation of the measure change in (2.1) and (2.2) for a general class of Lévy processes, which may be of independent interest. In Section 2.6.4, we show the natural result that a single component of $\mathbf{G}_n(\nu)$ or $\mathbf{M}_n(\nu)$ *conditioned to have size*

$\lfloor xn^{\alpha/(\alpha+1)} \rfloor$ has a component of the α-stable graph of size x as its scaling limit.

2.3 The limit object: the stable graph

2.3.1 An absolute continuity relation for spectrally positive α-stable Lévy processes

We begin by discussing the coding function R discovered by Joseph [91], which was defined in (2.5). Fix $\alpha \in (1, 2)$, $\mu \in (1, 2)$ and $c > 0$. Recall that L is the spectrally positive α-stable Lévy process having Lévy measure $\pi(dx) = \frac{c}{\mu} x^{-(\alpha+1)} dx$. This process has Laplace transform

$$\mathbb{E}\left[\exp(-\lambda L_t)\right] = \exp(t\Psi(\lambda)), \qquad \lambda \geq 0, \quad t \geq 0,$$

where

$$\Psi(\lambda) = \int_0^\infty \frac{c}{\mu} x^{-(\alpha+1)} dx (e^{-\lambda x} - 1 + \lambda x) = \frac{C_\alpha}{\mu} \lambda^\alpha,$$

with

$$C_\alpha = \frac{c\Gamma(2-\alpha)}{\alpha(\alpha-1)}.$$

Recall also that X is the unique process with independent increments such that

$$\mathbb{E}\left[\exp(-\lambda X_t)\right] = \exp\left(\int_0^t ds \int_0^\infty dx (e^{-\lambda x} - 1 + \lambda x) \frac{c}{\mu} \frac{1}{x^{\alpha+1}} e^{-xs/\mu} \right), \quad \lambda \geq 0, \quad t \geq 0.$$

Let

$$A_t = -C_\alpha \frac{t^\alpha}{\mu^\alpha}$$

and define

$$\widetilde{L}_t = X_t + A_t.$$

We observe that X is a martingale and A is a finite-variation process, so this is, in fact, the Doob–Meyer decomposition of the process $\widetilde{L}$.

Proposition 2.3.1. *We have*

$$\widetilde{L}_t \to -\infty \qquad a.s.$$

as $t \to \infty$.

Proof. Lemma B.3 of Joseph [91] gives the convergence in probability; we adapt his argument. Since A_t is deterministic and tends to $-\infty$, it will be sufficient to prove that

$$\limsup_{t\to\infty} t^{-\alpha} X_t = 0 \quad \text{a.s.}$$

Consider a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}_+$ of intensity

$$\frac{c}{\mu} x^{-(\alpha+1)} e^{-xs/\mu} \, ds \, dx,$$

with points $\{(s, \Delta_s)\}$. Let $X^{(1)}$ be the martingale arising from compensating the jumps of magnitude at most 1, formally defined as the $\epsilon \to 0$ limit of the family $\{X^{(1,\epsilon)}, \epsilon > 0\}$ of processes given by

$$X_t^{(1,\epsilon)} = \sum_{s \le t} \Delta_s \mathbb{1}_{\{\epsilon < \Delta_s < 1)\}} - \int_0^t ds \int_\epsilon^1 dx \frac{c}{\mu} x^{-\alpha} e^{-xs/\mu},$$

and let

$$X_t^{(2)} = \sum_{s \le t} \Delta_s \mathbb{1}_{\{\Delta_s \ge 1\}} - \int_0^t ds \int_1^\infty dx \frac{c}{\mu} x^{-\alpha} e^{-xs/\mu}.$$

Then $X_t = X_t^{(1)} + X_t^{(2)}$. By Doob's L^2-inequality, we have

$$\mathbb{E}\left[\left(\sup_{0 \le s \le t} \left|X_s^{(1)}\right|\right)^2\right] \le 4\mathbb{E}\left[(X_t^{(1)})^2\right] = 4 \int_0^t ds \int_0^1 dx \frac{c}{\mu} x^{-(\alpha-1)} e^{-xs/\mu}.$$

The integral on the right-hand side is bounded above by $Ct^{\alpha-1}$ for some constant $C > 0$. Indeed, for every $s > 0$,

$$\int_0^1 dx \frac{c}{\mu} x^{-(\alpha-1)} e^{-xs/\mu} = s^{\alpha-2} \int_0^{s/\mu} \frac{c}{\mu^{\alpha-1}} u^{-(\alpha-1)} e^{-u} du \le \frac{c\Gamma(2-\alpha)}{\mu^{\alpha-1}} s^{\alpha-2}.$$

Hence, applying Markov's inequality, we get

$$\mathbb{P}\left(\sup_{n-1 < s \le n} \left|X_s^{(1)}\right| > n^{(\alpha+1)/2}\right) \le \frac{C}{n^2}.$$

As this is summable in n, the Borel–Cantelli lemma gives that

$$\mathbb{P}\left(\sup_{n-1 < s \le n} \left|X_s^{(1)}\right| > n^{(\alpha+1)/2} \text{ i.o.}\right) = 0.$$

Since $\alpha > 1$, it follows that

$$\limsup_{t\to\infty} t^{-\alpha} X_t^{(1)} = 0 \quad \text{a.s.}$$

Turning now to $X^{(2)}$, for all $t \ge 0$ we have the straightforward bound

$$\sup_{t \ge 0} X_s^{(2)} \le \sum_{s \ge 0} \Delta_s \mathbb{1}_{\{\Delta_s \ge 1\}}.$$

The right-hand side has expectation

$$\int_0^\infty ds \int_1^\infty dx \frac{c}{\mu} x^{-\alpha} e^{-xs/\mu} = c \int_1^\infty dx\ x^{-(\alpha+1)} \int_0^\infty ds \frac{x}{\mu} e^{-xs/\mu} = c \int_1^\infty x^{-(\alpha+1)} dx = \frac{c}{\alpha}.$$

This computation also entails that

$$\inf_{t \ge 0} X_t^{(2)} \ge -\frac{c}{\alpha}.$$

Hence by Markov's inequality, we have for all $n \ge 1$:

$$\mathbb{P}\left(\sup_{n-1 < s \le n} \left|X_s^{(2)}\right| > n^{(\alpha+1)/2}\right) \le \frac{c}{\alpha n^{(\alpha+1)/2}}.$$

As for $X^{(1)}$, the Borel–Cantelli lemma gives that

$$\limsup_{t\to\infty} t^{-\alpha} X_t^{(2)} = 0 \quad \text{a.s.}$$

The result follows. □

The main purpose of this section is to expand considerably our understanding of the processes $\widetilde{L}$ and R. Our first new result says that the law of the process $\widetilde{L}$ is absolutely continuous with respect to the law of the Lévy process L on compact time-intervals.

Proposition 2.3.2. *For every $t \geq 0$, we have the following absolute continuity relation: for every non-negative integrable functional $f : \mathbb{D}([0,t],\mathbb{R}) \to \mathbb{R}_+$,*

$$\mathbb{E}\left[f(\widetilde{L}_s, 0 \leq s \leq t)\right] = \mathbb{E}\left[\exp\left(-\frac{1}{\mu}\int_0^t s dL_s - C_\alpha \frac{t^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right) f(L_s, 0 \leq s \leq t)\right].$$

This proposition is a consequence of a more general change of measure for spectrally positive Lévy processes, Proposition 2.6.2, which is proved in the Appendix below.

In the Brownian case, we instead have $L_t = \sqrt{\frac{\beta}{\mu}} B_t$, where B is a standard Brownian motion,

$$\widetilde{L}_t = \sqrt{\frac{\beta}{\mu}} B_t - \frac{\beta}{2\mu^2} t^2, \tag{2.11}$$

and Proposition 2.6.2 gives

$$\mathbb{E}\left[f(\widetilde{L}_s, 0 \leq s \leq t)\right] = \mathbb{E}\left[\exp\left(-\frac{1}{\mu}\int_0^t s dL_s - \frac{\beta}{6\mu^3} t^3\right) f\left(L_s, 0 \leq s \leq t\right)\right].$$

In order to harmonise notation, let us define $C_2 := \beta/2$, so that Proposition 2.3.2 is valid as stated for all $\alpha \in (1,2]$.

Remark 2.3.3. *The absolute continuity cannot be extended to $t = \infty$: the process $(L_t, t \geq 0)$ is recurrent whereas, by Proposition 2.3.1 for $\alpha \in (1,2)$ or (2.11) for $\alpha = 2$, we have $\widetilde{L}_t \to -\infty$ a.s. as $t \to \infty$. (In particular, $\left(\exp\left(-\frac{1}{\mu}\int_0^t s dL_s - C_\alpha \frac{t^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right), t \geq 0\right)$ is a martingale which is not uniformly integrable.)*

Recall that H is the height process which corresponds to L. Then for any $\alpha \in (1,2]$, we may define a pair $(\widetilde{L}, \widetilde{H})$ of processes via change of measure as follows: for suitable test-functions $f : \mathbb{D}([0,t],\mathbb{R})^2 \to \mathbb{R}$,

$$\mathbb{E}\left[f(\widetilde{L}_u, \widetilde{H}_u, 0 \leq u \leq t)\right] = \mathbb{E}\left[\exp\left(-\frac{1}{\mu}\int_0^t s dL_s - \frac{C_\alpha t^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right) f(L_u, H_u, 0 \leq u \leq t)\right].$$

2.3.2 Excursion theory

We begin with some notation. Write $\mathbb{D}_+(\mathbb{R}_+, \mathbb{R}_+)$ for the space of càdlàg functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ with only positive jumps. We write $\mathcal{E}$ for the space of excursions, that is

$$\mathcal{E} = \{\varepsilon \in \mathbb{D}_+(\mathbb{R}_+, \mathbb{R}_+) : \exists t > 0 \text{ s.t. } \varepsilon(s) > 0 \text{ for } s \in (0,t) \text{ and } \varepsilon(s) = 0 \text{ for } s \geq t\}.$$

For $\varepsilon \in \mathcal{E}$, let $\zeta(\varepsilon)$ be the lifetime of ε, that is the smallest t such that $\varepsilon(s) = 0$ for $s \geq t$. Let $\mathcal{E}^* = \mathcal{E} \cup \{\partial\}$, where the extra state, ∂, represents the empty excursion, with $\zeta(\partial) = 0$.

Let $I_t = \inf_{0 \le s \le t} L_s$. It is standard that the process $-I$ acts as a local time at 0 for the reflected Lévy process $(L_t - \inf_{0 \le s \le t} L_s, t \ge 0)$ (see Chapter VII of Bertoin [31] or Section 1.1.2 of Duquesne and Le Gall [70]). Indeed, we may decompose the path of the reflected process into excursions above 0. Now write

$$\sigma_\ell = \inf\{t \ge 0 : I_t < -\ell\},$$

so that $(\sigma_\ell, \ell \ge 0)$ is the inverse local time. We observe that σ_ℓ is a stopping time for the (usual augmentation of the) natural filtration of $(L_t)_{t \ge 0}$. For $\ell \ge 0$, write

$$\varepsilon^{(\ell)} = \begin{cases} (\ell + L_{\sigma_{\ell-}+u}, 0 \le u \le \sigma_\ell - \sigma_{\ell-}) & \text{if } \sigma_\ell - \sigma_{\ell-} > 0 \\ \partial & \text{otherwise.} \end{cases}$$

Then the following theorem is standard (see Theorem VII.1.1 of Bertoin [31], converting from the spectrally negative case, or Miermont [109] for a convenient reference).

Theorem 2.3.4. *The inverse local time process $(\sigma_\ell, \ell \ge 0)$ is a stable subordinator of index $1/\alpha$ and, more specifically, with Lévy measure*

$$\frac{\mu^{1/\alpha}}{C_\alpha^{1/\alpha} \alpha \Gamma(1 - 1/\alpha)} x^{-1-1/\alpha} dx.$$

Moreover, the point measure on $\mathbb{R}_+ \times \mathcal{E}$ given by

$$\sum_{s \ge 0 : \sigma_s - \sigma_{s-} > 0} \delta_{(s, \varepsilon^{(s)})} \tag{2.12}$$

is a Poisson random measure of intensity $d\ell \otimes \mathbb{N}(d\mathbb{e})$, where the excursion measure $\mathbb{N}$ satisfies

$$\mathbb{N}(\zeta(\mathbb{e}) \in dx) = \frac{\mu^{1/\alpha}}{C_\alpha^{1/\alpha} \alpha \Gamma(1 - 1/\alpha)} x^{-1-1/\alpha} dx.$$

Consider the excursions occurring before time σ_ℓ. With probability 1, only finitely many of these are longer than η in duration for any $\eta > 0$. So, in particular, they may be listed in decreasing order of length as $(\varepsilon_i^{(\ell)}, i \ge 1)$.

Since L is self-similar, it is possible to make sense of normalised versions of $\mathbb{N}$ i.e. $\mathbb{N}^{(x)}(\cdot) = \mathbb{N}(\cdot | \zeta(\mathbb{e}) = x)$, which are probability measures. (Again see Miermont [109] for more details.) For example, the law of $\mathbb{e}$ under $\mathbb{N}^{(x)}$ is the same as the law of

$$\Big((x/\zeta(\mathbb{e}))^{1/\alpha} \mathbb{e}(\zeta(\mathbb{e}) s / x), 0 \le s \le x \Big)$$

under $\mathbb{N}(\cdot | \zeta(\mathbb{e}) > \eta)$ for any fixed $\eta > 0$. In particular, we have that under $\mathbb{N}^{(x)}$, the rescaled excursion $(x^{-1/\alpha} \mathbb{e}(xu), 0 \le u \le 1)$ has the same law as $\mathbb{e}$ under $\mathbb{N}^{(1)}$. It follows that the excursions $\varepsilon^{(s)}$ appearing in (2.12) may be thought of in two parts: as their lengths $\zeta(\varepsilon^{(s)}) = \sigma_s - \sigma_{s-}$ and their normalised "shapes" $e^{(s)} := \big(\zeta(\varepsilon^{(s)})^{-1/\alpha} \varepsilon^{(s)}(\zeta(\varepsilon^{(s)}) u), 0 \le u \le 1 \big)$ where, crucially, the collection of shapes $(e^{(s)}, s \geq 0)$ is *independent* of the collection of excursion lengths $(\zeta(\varepsilon^{(s)}), s \geq 0)$.

We observe that the excursions of the Lévy process L above its running infimum and the excursions of the height process H are in one-to-one correspondence and have the same lengths. In particular, we can make sense of an excursion of the height process $\mathbb{h}$ derived from $\mathbb{e}$, under $\mathbb{N}$ or its conditioned versions. The scaling relation for the height process is that under $\mathbb{N}^{(x)}$ the rescaled excursion $(x^{-(\alpha-1)/\alpha}\mathbb{h}(xu), 0 \leq u \leq 1)$ has the same law as $\mathbb{h}$ under $\mathbb{N}^{(1)}$. The usual *stable tree* is encoded by (a scalar multiple of) a height process with the distribution of $\mathbb{h}$ under $\mathbb{N}^{(1)}$.

Much of this structure can be transferred into our setting, by absolute continuity. Recall that

$$R_t = \widetilde{L}_t - \inf_{0 \leq s \leq t} \widetilde{L}_s, \quad t \geq 0.$$

We will make use of the following properties.

Lemma 2.3.5. *The following statements hold almost surely.*

(i) For each $\epsilon > 0$, R has only finitely many excursions of length greater than or equal to ϵ.

(ii) The set $\{t : R_t = 0\}$ has Lebesgue measure 0.

(iii) If (l_1, r_1) and (l_2, r_2) are excursion-intervals of R and $l_1 < l_2$, then $\widetilde{L}_{l_1} > \widetilde{L}_{l_2}$.

(iv) For $a \geq 0$, let $\mathcal{B}_a = \{b > a : \widetilde{L}_{b-} = \inf_{a \leq s \leq b} \widetilde{L}_s\}$. Then $\mathcal{B}_a$ does not intersect the set of jump times of $\widetilde{L}$.

Proof. Part (i) is a consequence of Lemma B.3 of Joseph [91]. For parts (ii), (iii) and (iv), we first argue that the claimed properties are almost surely true for the Lévy process L and then use absolute continuity to deduce them for $\widetilde{L}$.

The analogues of both (ii) and (iii) are standard for L (see, for example, Chapter VII of Bertoin [31]; indeed, these properties are necessary for Theorem 2.3.4 to work). It follows by absolute continuity that $\mathbb{P}(\text{Leb}(\{s \leq t : R_s = 0\}) = 0) = 1$ and

$$\mathbb{P}\left(\widetilde{L}_{l_1} > \widetilde{L}_{l_2} \text{ for all } (l_1, r_1), (l_2, r_2) \text{ excursion-intervals of } R \text{ with } l_1 < l_2 \leq t\right) = 1,$$

for fixed $t > 0$. But then (ii) and (iii) follow by monotone convergence.

By the stationarity and independence of increments of L, it is sufficient to prove (iv) for $a = 0$. But this then follows from Corollary 1 of Rogers [124]. In particular, if we let $\mathcal{J}$ be the set of jump-times of $\widetilde{L}$, by absolute continuity we get $\mathbb{P}(\mathcal{B}_a \cap [0, t] \cap \mathcal{J} \neq \emptyset) = 0$ for any $t > 0$. By monotone convergence again, we obtain $\mathbb{P}(\mathcal{B}_a \cap \mathcal{J} \neq \emptyset) = 0$. □

Let $\widetilde{I}_t = \inf_{0 \leq s \leq t} \widetilde{L}_s$. As for the reflected stable process, we have that $-\widetilde{I}$ acts as a local time at 0 for R. We write $(\widetilde{\sigma}_\ell, \ell \geq 0)$ with $\widetilde{\sigma}_\ell = \inf\{t > 0 : \widetilde{I}_t < -\ell\}$ for the inverse local time, $(\widetilde{\varepsilon}^{(\ell)}, \ell \geq 0)$ for the collection of excursions above 0, indexed by local time (with $\widetilde{\varepsilon}^{(\ell)} = \partial$ if $\widetilde{\sigma}_\ell - \widetilde{\sigma}_{\ell-} = 0$), and $(\widetilde{e}^{(\ell)}, \ell \geq 0)$ for their shapes. In order to understand the laws of these quantities, we first need to prove two preliminary results, Lemma 2.3.6 and Proposition 2.3.7.

Lemma 2.3.6. *Let $\alpha \in (1, 2]$. Then for any $\theta > 0$,*

$$\mathbb{E}_{\mathbb{N}^{(1)}}\left[\exp\left(\theta \int_0^1 \mathbb{e}(t)dt\right)\right] < \infty.$$

Proof. This is well known in the $\alpha = 2$ case; see, for example, Section 13 of Janson [88]. For $\alpha \in (1, 2)$, observe that

$$\int_0^1 \mathbb{e}(t)dt \leq \sup_{t\in[0,1]} \mathbb{e}(t).$$

By Theorem 9 of Kortchemski [92] (see also the discussion at the top of the 12th page), for any $\delta \in (0, \frac{\alpha}{\alpha-1})$, there exist constants $C_1, C_2 > 0$ such that

$$\mathbb{E}_{\mathbb{N}^{(1)}}\left(\sup_{t\in[0,1]} \mathbb{e}(t) \geq u\right) \leq C_1 \exp(-C_2 u^\delta),$$

for every $u \geq 0$. (Note that since Kortchemski works with the Lévy process having Laplace exponent λ^α, his normalised excursions are a constant scaling factor different from ours. But this changes the bound only by a constant.) Since we may take $\delta > 1$, the result follows. □

For $t \geq 0$ write

$$\Phi(t) := \exp\left(-\frac{1}{\mu}\int_0^t sdL_s - \frac{C_\alpha t^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right).$$

Proposition 2.3.7. *Let $\alpha \in (1, 2]$. For any $\ell \geq 0$ we have that $(\Phi(t \wedge \sigma_\ell), t \geq 0)$ is a uniformly integrable martingale and, thus, $\mathbb{E}[\Phi(\sigma_\ell)] = 1$.*

Proof. By Lemma 2.6.1 with $\theta = \frac{1}{\mu}$, $\gamma = \delta = 0$ and $\pi(dx) = \frac{c}{\mu}x^{-(\alpha+1)}dx$, $(\Phi(t), t \geq 0)$ is a non-negative martingale of mean 1. Since for any $\ell \geq 0$, σ_ℓ is a stopping time for L and since Φ has right-continuous trajectories, $(\Phi(t \wedge \sigma_\ell), t \geq 0)$ is a martingale (w.r.t to the natural filtration of L). So by the almost sure martingale convergence theorem, we must have $\Phi(t \wedge \sigma_\ell) \to \Phi(\sigma_\ell)$ almost surely as $t \to \infty$. Then $(\Phi(t \wedge \sigma_\ell), t \geq 0)$ is uniformly integrable if and only if this convergence also holds in L^1. By Fatou's lemma, we get $\mathbb{E}[\Phi(\sigma_\ell)] \leq 1$, so that $\Phi(\sigma_\ell)$ is integrable. Now, for any $t > 0$,

$$\mathbb{E}[|\Phi(\sigma_\ell) - \Phi(t \wedge \sigma_\ell)|] = \mathbb{E}\left[|\Phi(\sigma_\ell) - \Phi(t)|\mathbb{1}_{\{\sigma_\ell > t\}}\right] \leq \mathbb{E}\left[\Phi(\sigma_\ell)\mathbb{1}_{\{\sigma_\ell > t\}}\right] + \mathbb{E}\left[\Phi(t)\mathbb{1}_{\{\sigma_\ell > t\}}\right].$$

Observe that by the definition of the measure-changed process, we have $\mathbb{E}\left[\Phi(t)\mathbb{1}_{\{\sigma_\ell > t\}}\right] = \mathbb{P}(\widetilde{\sigma}_\ell > t)$. So

$$\mathbb{E}[|\Phi(\sigma_\ell) - \Phi(t \wedge \sigma_\ell)|] \leq \mathbb{P}(\widetilde{\sigma}_\ell > t) + \mathbb{E}\left[\Phi(\sigma_\ell)\mathbb{1}_{\{\sigma_\ell > t\}}\right].$$

Since $\Phi(\sigma_\ell)$ is integrable (and hence uniformly integrable) and since $\sigma_\ell < \infty$ almost surely, we have $\lim_{t\to\infty} \mathbb{E}\left[\Phi(\sigma_\ell)\mathbb{1}_{\{\sigma_\ell > t\}}\right] = 0$. By Proposition 2.3.1, we have that $\widetilde{L}_t \to -\infty$ almost surely as $t \to \infty$, and so $\widetilde{\sigma}_\ell < \infty$ almost surely. So $\lim_{t\to\infty} \mathbb{P}\left(\widetilde{\sigma}_\ell > t\right) = 0$ and we get

$$\mathbb{E}\left[|\Phi(\sigma_\ell) - \Phi(t \wedge \sigma_\ell)|\right] \to 0$$

as $t \to \infty$. Hence, $(\Phi(t \wedge \sigma_\ell), t \geq 0)$ is uniformly integrable and, in particular, we may deduce that $\mathbb{E}\left[\Phi(\sigma_\ell)\right] = 1$. □

We are now in a position to characterise the joint law of $(\widetilde{\sigma}_s, 0 \leq s \leq \ell)$ and $(\widetilde{\varepsilon}^{(s)}, s \leq \ell)$. We will find it convenient to list the excursions occurring before local time ℓ has been accumulated in decreasing order of length, as $(\widetilde{\varepsilon}_i^{(\ell)}, i \geq 1)$. Proposition 2.3.7 implies that we may use the Radon–Nikodym derivative $\Phi(t)$ to change measure at the *random* times σ_ℓ. As earlier, we write $(\varepsilon_i^{(\ell)}, i \geq 1)$ for the excursions of L occurring before time σ_ℓ in decreasing order of length. For an excursion $\varepsilon \in \mathcal{E}^* = \mathcal{E} \cup \{\partial\}$, write $a(\varepsilon) = \int_0^{\zeta(\varepsilon)} \varepsilon(u)du$ for its area.

Proposition 2.3.8. *For suitable test functions f and $g_1, g_2, g_3, \ldots$, and any $n \geq 1$, we have*

$$\begin{aligned}
&\mathbb{E}\left[f(\widetilde{\sigma}_s, 0 \leq s \leq \ell)\prod_{i=1}^n g_i\left(\widetilde{\varepsilon}_i^{(\ell)}\right)\right] \\
&= \mathbb{E}\left[\exp\left(\frac{1}{\mu}\int_0^\ell \sigma_r dr - \frac{C_\alpha \sigma_\ell^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right) f(\sigma_s, 0 \leq s \leq \ell)\right. \\
&\quad \left.\times \mathbb{E}\left[\exp\left(\frac{1}{\mu}\sum_{j>n} a(\varepsilon_j^{(\ell)})\right) \middle| \zeta(\varepsilon_k^{(\ell)}), k > n\right] \prod_{i=1}^n \mathbb{E}\left[\exp\left(\frac{1}{\mu}a(\varepsilon_i^{(\ell)})\right) g_i\left(\varepsilon_i^{(\ell)}\right) \middle| \zeta(\varepsilon_i^{(\ell)})\right]\right].
\end{aligned}$$

In particular, the excursions $(\widetilde{\varepsilon}_i^{(\ell)}, i \geq 1)$ are conditionally independent given their lengths. Moreover, for any $i \geq 1$ and any suitable test function g,

$$\begin{aligned}
\mathbb{E}\left[g\left(\widetilde{\varepsilon}_i^{(\ell)}\right) \;\middle|\; \zeta(\widetilde{\varepsilon}_i^{(\ell)}) = x\right] &= \frac{\mathbb{E}_{\mathbb{N}^{(x)}}\left[\exp\left(\frac{1}{\mu}\int_0^x \mathbb{e}(t)dt\right) g(\mathbb{e})\right]}{\mathbb{E}_{\mathbb{N}^{(x)}}\left[\exp\left(\frac{1}{\mu}\int_0^x \mathbb{e}(t)dt\right)\right]} \\
&= \frac{\mathbb{E}_{\mathbb{N}^{(1)}}\left[\exp\left(\frac{x^{1+1/\alpha}}{\mu}\int_0^1 \mathbb{e}(t)dt\right) g(x^{1/\alpha}\mathbb{e}(\cdot/x))\right]}{\mathbb{E}_{\mathbb{N}^{(1)}}\left[\exp\left(\frac{x^{1+1/\alpha}}{\mu}\int_0^1 \mathbb{e}(t)dt\right)\right]}.
\end{aligned}$$

Proof. By integration by parts and writing $L_s = I_s + (L_s - I_s)$, noting that $L_{\sigma_\ell} = -\ell$, we get

$$-\frac{1}{\mu}\int_0^{\sigma_\ell} s dL_s = \frac{\ell\sigma_\ell}{\mu} + \frac{1}{\mu}\int_0^{\sigma_\ell} L_s ds = \frac{\ell\sigma_\ell}{\mu} + \frac{1}{\mu}\int_0^{\sigma_\ell} I_s ds + \frac{1}{\mu}\int_0^{\sigma_\ell}(L_s - I_s)ds.$$

Changing variable in the middle term, and using the fact that $I_{\sigma_s} = -s$, we obtain

$$\frac{\ell\sigma_\ell}{\mu} + \frac{1}{\mu}\int_0^\ell I_{\sigma_s} d\sigma_s + \frac{1}{\mu}\int_0^{\sigma_\ell}(L_s - I_s)ds = \frac{\ell\sigma_\ell}{\mu} - \frac{1}{\mu}\int_0^\ell s d\sigma_s + \frac{1}{\mu}\int_0^{\sigma_\ell}(L_s - I_s)ds.$$

Another integration by parts yields that this is equal to

$$\frac{1}{\mu}\int_0^\ell \sigma_s ds + \frac{1}{\mu}\int_0^{\sigma_\ell}(L_s - I_s)ds.$$

Finally, we can integrate the excursions of $L - I$ separately to obtain that this is equal to

$$\frac{1}{\mu}\int_0^\ell \sigma_s ds + \frac{1}{\mu}\sum_{s\le\ell} a(\varepsilon^{(s)}).$$

Hence,

$$\Phi(\sigma_\ell) = \exp\left(\frac{1}{\mu}\int_0^\ell \sigma_r dr + \frac{1}{\mu}\sum_{s\le\ell} a(\varepsilon^{(s)}) - C_\alpha \frac{\sigma_\ell^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right). \tag{2.13}$$

Now,

$$\begin{aligned}
&\mathbb{E}\left[f(\widetilde{\sigma}_s, 0\le s\le\ell)\prod_{i=1}^n g_i\left(\widetilde{\varepsilon}_i^{(\ell)}\right)\right] \\
&= \mathbb{E}\left[\exp\left(\frac{1}{\mu}\int_0^\ell \sigma_r dr + \frac{1}{\mu}\sum_{s\le\ell} a(\varepsilon^{(s)}) - C_\alpha \frac{\sigma_\ell^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right) f(\sigma_s, 0\le s\le\ell)\prod_{i=1}^n g_i\left(\varepsilon_i^{(\ell)}\right)\right] \\
&= \mathbb{E}\left[\exp\left(\frac{1}{\mu}\int_0^\ell \sigma_r dr - C_\alpha \frac{\sigma_\ell^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right) f(\sigma_s, 0\le s\le\ell)\right. \\
&\qquad \left.\times\mathbb{E}\left[\exp\left(\frac{1}{\mu}\sum_{s\le\ell} a(\varepsilon^{(s)})\right)\prod_{i=1}^n f_i\left(\varepsilon_i^{(\ell)}\right)\Big|(\sigma_s, 0\le s\le\ell)\right]\right],
\end{aligned}$$

As discussed below Theorem 2.3.4, the excursions of the stable Lévy process are conditionally independent given their lengths, which yields the first expression in the statement of the proposition. The final statement is an immediate consequence of the scaling property for stable excursions; we observe that this change of measure for the excursions is well-defined by Lemma 2.3.6. □

Lemma 2.3.5 (i) implies that we can list *all* the excursions of R in decreasing order of length: write $(\widetilde{\varepsilon}_i, i\ge 1)$ for this list. Write $(\widetilde{h}_i, i\ge 1)$ for the corresponding height process excursions.

Proposition 2.3.9. *The pairs of excursions $(\widetilde{\varepsilon}_i, \widetilde{h}_i, i\ge 1)$ are conditionally independent given their lengths $(\zeta(\widetilde{\varepsilon}_i), i\ge 1)$, with law specified by*

$$\begin{aligned}
\mathbb{E}\left[g\left(\widetilde{\varepsilon}_i, \widetilde{h}_i\right)\ \Big|\ \zeta(\widetilde{\varepsilon}_i) = x\right] &= \frac{\mathbb{E}_{\mathbb{N}^{(x)}}\left[\exp\left(\frac{1}{\mu}\int_0^x \mathbb{e}(t)dt\right) g(\mathbb{e}, \mathbb{h})\right]}{\mathbb{E}_{\mathbb{N}^{(x)}}\left[\exp\left(\frac{1}{\mu}\int_0^x \mathbb{e}(t)dt\right)\right]} \\
&= \frac{\mathbb{E}_{\mathbb{N}^{(1)}}\left[\exp\left(\frac{x^{1+1/\alpha}}{\mu}\int_0^1 \mathbb{e}(t)dt\right) g(x^{1/\alpha}\mathbb{e}(\cdot/x), x^{(\alpha-1)/\alpha}\mathbb{h}(\cdot/x))\right]}{\mathbb{E}_{\mathbb{N}^{(1)}}\left[\exp\left(\frac{x^{1+1/\alpha}}{\mu}\int_0^1 \mathbb{e}(t)dt\right)\right]}.
\end{aligned}$$

Proof. The excursions of R occurring before local time ℓ has been accumulated are a strict subset of all the excursions that ever occur. By Lemma 2.3.5, we have that

$$\sup\{\zeta(\varepsilon) :\ \varepsilon \text{ is an excursion of } R \text{ starting after time } t\} \xrightarrow{p} 0$$

as $t \to \infty$ and $-\widetilde{I}_t \to \infty$ as $t \to \infty$. The latter implies that $\widetilde{\sigma}_\ell < \infty$ a.s., and since $-\widetilde{I}_t < \infty$ for each $t > 0$, we also have $\widetilde{\sigma}_\ell \to \infty$ as $\ell \to \infty$. Hence,

$$\sup\{\zeta(\varepsilon) : \ \varepsilon \text{ is an excursion of } R \text{ starting after time } \widetilde{\sigma}_\ell\} \xrightarrow{p} 0$$

as $\ell \to \infty$. It follows that

$$(\zeta(\widetilde{\varepsilon}_i^{(\ell)}), i \geq 1) \to (\zeta(\widetilde{\varepsilon}_i), i \geq 1) \quad \text{a.s.}$$

in the product topology, as $\ell \to \infty$. The result then follows from Proposition 2.3.8 since the expressions there do not depend on the value of ℓ. □

This enables us to give the proof of Theorem 2.1.2 assuming the definition of $((\mathcal{G}_i, d_i, \mu_i), i \geq 1)$ from $\widetilde{L}$ given following Theorem 2.1.1.

Proof of Theorem 2.1.2. With Proposition 2.3.9 in hand, it remains to deal with the Poisson points which give rise to the vertex-identifications. We have straightforwardly that, given $\widetilde{\varepsilon}_i$, the number M_i of points falling under the excursion is conditionally independent of the other excursions and has a Poisson distribution with parameter $\frac{1}{\mu}\int_0^\infty \widetilde{\varepsilon}_i(u)du$. Moreover, conditionally on the number of points, their locations are i.i.d. uniform random variables in the area under the excursion. For any suitable test function g,

$$\begin{aligned}
&\mathbb{E}\left[g\left(\widetilde{\varepsilon}_i, \widetilde{h}_i\right) \mathbb{1}_{\{M_i = m\}} \,\middle|\, \zeta(\widetilde{\varepsilon}_i) = x\right] \\
&\quad = \mathbb{E}\left[g\left(\widetilde{\varepsilon}_i, \widetilde{h}_i\right) \exp\left(-\frac{1}{\mu}\int_0^\infty \widetilde{\varepsilon}_i(u)du\right) \frac{1}{m!}\left(\frac{1}{\mu}\int_0^\infty \widetilde{\varepsilon}_i(u)du\right)^m \,\middle|\, \zeta(\widetilde{\varepsilon}_i) = x\right] \\
&\quad = \frac{\mathbb{E}_{\mathbb{N}^{(x)}}\left[\exp\left(\frac{1}{\mu}\int_0^x \mathbb{e}(t)dt\right) g(\mathbb{e}, \mathbb{h}) \exp\left(-\frac{1}{\mu}\int_0^x \mathbb{e}(t)dt\right) \frac{1}{m!}\left(\frac{1}{\mu}\int_0^x \mathbb{e}(t)dt\right)^m\right]}{\mathbb{E}_{\mathbb{N}^{(x)}}\left[\exp\left(\frac{1}{\mu}\int_0^x \mathbb{e}(t)dt\right)\right]}
\end{aligned}$$

and so

$$\begin{aligned}
\mathbb{E}\left[g\left(\widetilde{\varepsilon}_i, \widetilde{h}_i\right) \,\middle|\, \zeta(\widetilde{\varepsilon}_i) = x, M_i = m\right] &= \frac{\mathbb{E}_{\mathbb{N}^{(x)}}\left[\left(\frac{1}{\mu}\int_0^x \mathbb{e}(t)dt\right)^m g(\mathbb{e}, \mathbb{h})\right]}{\mathbb{E}_{\mathbb{N}^{(x)}}\left[\left(\frac{1}{\mu}\int_0^x \mathbb{e}(t)dt\right)^m\right]} \\
&= \frac{\mathbb{E}_{\mathbb{N}^{(1)}}\left[\left(\int_0^1 \mathbb{e}(t)dt\right)^m g(x^{1/\alpha}\mathbb{e}(\cdot/x), x^{(\alpha-1)/\alpha}\mathbb{h}(\cdot/x))\right]}{\mathbb{E}_{\mathbb{N}^{(1)}}\left[\left(\int_0^1 \mathbb{e}(t)dt\right)^m\right]}.
\end{aligned}$$

The claimed result follows. □

2.4 Convergence of a discrete forest

The multigraph $\mathbf{M}_n(\nu)$ contains cycles with probability tending to 1 as $n \to \infty$. However, its components will turn out to be tree-like, in that they each have a finite surplus, with probability

1. In this section, we study an idealised version of the depth-first walk of the multigraph, ignoring cycles.

Let $(\hat{D}_1^n, \hat{D}_2^n, \dots, \hat{D}_k^n)$ be $D_1, D_2, \dots, D_n$ arranged in size-biased random order. More precisely, let Σ be a random permutation of $\{1, 2, \dots, n\}$ such that

$$\mathbb{P}(\Sigma = \sigma | D_1, \dots, D_n) = \frac{D_{\sigma(1)}}{\sum_{j=1}^n D_{\sigma(j)}} \frac{D_{\sigma(2)}}{\sum_{j=2}^n D_{\sigma(j)}} \cdots \frac{D_{\sigma(n)}}{D_{\sigma(n)}}$$

and define

$$(\hat{D}_1^n, \hat{D}_2^n, \dots, \hat{D}_n^n) = (D_{\Sigma(1)}, D_{\Sigma(2)}, \dots, D_{\Sigma(n)}).$$

Now let $\widetilde{S}^n(0) = 0$ and, for $k \geq 1$,

$$\widetilde{S}^n(k) = \sum_{i=1}^k (\hat{D}_i^n - 2).$$

Then $\widetilde{S}^n$ is the depth-first walk of a forest of trees in which the ith vertex visited in depth-first order has $\hat{D}_i^n - 1 \geq 0$ children. Define the corresponding height process,

$$\widetilde{G}^n(k) = \# \left\{ j \in \{0, 1, \dots, k-1\} : \widetilde{S}^n(j) = \inf_{j \leq \ell \leq k} \widetilde{S}^n(\ell) \right\}.$$

The purpose of this section is to recover Theorem 8.1 of Joseph [91] and, indeed, to strengthen it by adding the convergence of the height process to that of the depth-first walk. We will prove the following.

Theorem 2.4.1. *We have*

$$\left(n^{-\frac{1}{\alpha+1}} \widetilde{S}^n(\lfloor n^{\frac{\alpha}{\alpha+1}} t \rfloor), n^{-\frac{\alpha-1}{\alpha+1}} \widetilde{G}^n(\lfloor n^{\frac{\alpha}{\alpha+1}} t \rfloor), t \geq 0 \right) \xrightarrow{d} (\widetilde{L}_t, \widetilde{H}_t, t \geq 0)$$

as $n \to \infty$ in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})^2$.

In order to prove this theorem, we will begin by showing that there is an analogue in the discrete setting of the change of measure used to define $\widetilde{L}$.

Write $Z_1, Z_2, \dots, Z_n$ for i.i.d. random variables with the size-biased degree distribution, i.e.

$$\mathbb{P}(Z_1 = k) = \frac{k \nu_k}{\mu}, \quad k \geq 1.$$

Observe that $\mu \in (1, 2)$ since, firstly, $D_1 \geq 1$ and, secondly, $\mathbb{E}[D_1^2] = 2\mu$ and we must have $\mathrm{var}(D_1) = \mu(2 - \mu) > 0$. Then we have $\mathbb{E}[Z_1] = 2$, $\mathbb{P}(Z_1 \geq 1) = 1$ and $\mathbb{P}(Z_1 = k) \sim \frac{c}{\mu} k^{-(\alpha+1)}$ as $k \to \infty$ if $\alpha \in (1, 2)$, or $\mathrm{var}(Z_1) = \beta/\mu$ if $\alpha = 2$.

Proposition 2.4.2. *For any $k_1, k_2, \dots, k_n \geq 1$, we have*

$$\mathbb{P}\left(\hat{D}_1^n = k_1, \hat{D}_2^n = k_2, \dots, \hat{D}_n^n = k_n \right) = k_1 \nu_{k_1} k_2 \nu_{k_2} \dots k_n \nu_{k_n} \prod_{i=1}^n \frac{(n - i + 1)}{\sum_{j=i}^n k_j}.$$

Moreover, for $0 \le m \le n$ *and* $k_1, k_2, \ldots, k_m \ge 1$, *let*

$$\phi_m^n(k_1, k_2, \ldots, k_m) := \mathbb{E}\left[\prod_{i=1}^m \frac{(n-i+1)\mu}{\sum_{j=i}^m k_j + \Xi_{n-m}}\right],$$

where Ξ_{n-m} *has the same law as* $D_{m+1} + D_{m+2} + \cdots + D_n$. *Then for any suitable test-function* $g : Z_+^m \to \mathbb{R}_+$,

$$\mathbb{E}\left[g(\hat{D}_1^n, \hat{D}_2^n, \ldots, \hat{D}_m^n)\right] = \mathbb{E}\left[\phi_m^n(Z_1, Z_2, \ldots, Z_m) g(Z_1, Z_2, \ldots, Z_m)\right]. \tag{2.14}$$

We have not found a precise reference for the contents of Proposition 2.4.2. The analogue of (2.14) for continuous random variables is equation (1) of Barouch & Kaufman [25]; see Proposition 1 of Pitman & Tran [120] for a proof. The proof of Proposition 2.4.2 is elementary and may be found in the Appendix.

We now show that the Radon–Nikodym derivative in the change-of-measure formula converges in distribution under appropriate conditions. Until the end of this section, we restrict our attention to the case $\alpha \in (1, 2)$; the proof for the Brownian case is similar but a little more involved, so we defer it to Section 2.6.3 in the Appendix.

Proposition 2.4.3. *Let*

$$\Phi(n, m) := \phi_m^n(Z_1, Z_2, \ldots, Z_m)$$

and recall that

$$\Phi(t) = \exp\left(-\frac{1}{\mu}\int_0^t s dL_s - \frac{C_\alpha t^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right).$$

Then for fixed $t > 0$, $\Phi(n, \lfloor tn^{\frac{\alpha}{\alpha+1}} \rfloor) \xrightarrow{d} \Phi(t)$ *as* $n \to \infty$. *Moreover, the sequence of random variables* $(\Phi(n, \lfloor tn^{\frac{\alpha}{\alpha+1}} \rfloor))_{n \ge 1}$ *is uniformly integrable.*

In order to prove this, we will need some technical lemmas.

First, we consider the asymptotics of $S(k) = \sum_{i=1}^k (Z_i - 2)$. The generalised functional central limit theorem, Theorem 2.2.3 (ii), entails that

$$n^{-1/(\alpha+1)}\left(S\left(\lfloor tn^{\alpha/(\alpha+1)} \rfloor\right), t \ge 0\right) \xrightarrow{d} (L_t, t \ge 0) \tag{2.15}$$

as $n \to \infty$ in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, where L is the spectrally positive α-stable Lévy process introduced in the previous section. We will need to deal with functionals of S converging, which we will do via the continuous mapping theorem (Theorem 3.2.4 of Durrett [71]). We give here the details for the functional which will arise most frequently in the sequel.

Lemma 2.4.4. *For any* $t \ge 0$,

$$\frac{1}{n}\sum_{k=0}^{\lfloor tn^{\frac{\alpha}{\alpha+1}} \rfloor - 1} S(k) \xrightarrow{d} \int_0^t L_s ds$$

as $n \to \infty$.

Proof. We have

$$\frac{1}{n}\sum_{k=0}^{\lfloor tn^{\frac{\alpha}{\alpha+1}}\rfloor-1} S(k) = \frac{1}{n}\int_0^{\lfloor tn^{\frac{\alpha}{\alpha+1}}\rfloor} S(\lfloor v\rfloor)dv = \int_0^{n^{-\frac{\alpha}{\alpha+1}}\lfloor tn^{\frac{\alpha}{\alpha+1}}\rfloor} n^{-\frac{1}{\alpha+1}} S(\lfloor un^{\frac{\alpha}{\alpha+1}}\rfloor)du,$$

by changing variable. Then the convergence in law follows from the fact that $n^{-\frac{\alpha}{\alpha+1}}\lfloor tn^{\frac{\alpha}{\alpha+1}}\rfloor \to t$ and the continuous mapping theorem. □

Lemma 2.4.5. *Let $\mathcal{L}(\lambda) := \mathbb{E}\left[\exp(-\lambda D_1)\right]$. Then as $\lambda \to 0$,*

$$\mathcal{L}(\lambda) = \exp\left(-\mu\lambda + \frac{\mu(2-\mu)}{2}\lambda^2 - \frac{C_\alpha \lambda^{\alpha+1}}{(\alpha+1)} + o(\lambda^{\alpha+1})\right). \tag{2.16}$$

Proof. First observe that

$$\mathcal{L}'''(\lambda) = -\mathbb{E}\left[D_1^3 \exp(-\lambda D_1)\right] = -\sum_{k=1}^{\infty} k^3 e^{-\lambda k}\nu_k.$$

Since $\nu_k \sim ck^{-(\alpha+2)}$, the right-hand side is finite and, by the Euler–Maclaurin formula, asymptotically equivalent to

$$\int_0^\infty cx^{1-\alpha}e^{-\lambda x}dx = c\lambda^{\alpha-2}\Gamma(2-\alpha),$$

as $\lambda \to 0$. In other words,

$$\mathcal{L}'''(\lambda) = -c\lambda^{\alpha-2}\Gamma(2-\alpha) + o(\lambda^{\alpha-2}),$$

where o is for $\lambda \to 0$. We also have $\mathbb{E}\left[D_1\right] = \mu$ and $\mathbb{E}\left[D_1^2\right] = 2\mu$. So integrating three times, we obtain

$$\mathcal{L}(\lambda) = 1 - \mu\lambda + \mu\lambda^2 - \frac{c\Gamma(2-\alpha)}{(\alpha-1)\alpha(\alpha+1)}\lambda^{\alpha+1} + o(\lambda^{\alpha+1}),$$

and it is straightforward to see that this implies

$$\mathcal{L}(\lambda) = \exp\left(-\mu\lambda + \frac{\mu(2-\mu)}{2}\lambda^2 - \frac{C_\alpha \lambda^{\alpha+1}}{(\alpha+1)} + o(\lambda^{\alpha+1})\right). \qquad \square$$

Lemma 2.4.6. *For $m = O(n^{\alpha/(\alpha+1)})$, we have*

$$\exp\left(m - \frac{(2+\mu)}{2\mu}\frac{m^2}{n}\right)\left[\mathcal{L}\left(\frac{m}{n\mu}\right)\right]^{n-m} = (1+o(1))\exp\left(-\frac{C_\alpha m^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}n^\alpha}\right).$$

Proof. By (2.16), it is sufficient to show that

$$\begin{aligned} m - \frac{(2+\mu)}{2\mu}\frac{m^2}{n} + (n-m)\left(-\frac{\mu m}{n\mu} + \frac{\mu(2-\mu)}{2}\frac{m^2}{n^2\mu^2} - \frac{C_\alpha}{(\alpha+1)}\frac{m^{\alpha+1}}{n^{\alpha+1}\mu^{\alpha+1}}\right) \\ = -\frac{C_\alpha m^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}n^\alpha} + o(1), \end{aligned}$$

as $n \to \infty$. But this is now easily seen to be true on cancellation and using $m = O(n^{\alpha/(\alpha+1)})$. The result follows. □

Lemma 2.4.7. *Let $s(0) = 0$ and $s(i) = \sum_{j=1}^{i}(k_j - 2)$ for $i \geq 1$. Then if $m = O(n^{\alpha/(\alpha+1)})$, we have*

$$\phi_n^m(k_1, k_2, \ldots, k_m) \geq \exp\left(\frac{1}{n\mu}\sum_{i=0}^{m}(s(i) - s(m)) - \frac{C_\alpha m^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}n^\alpha}\right)(1 + o(1)),$$

where the $o(1)$ term is independent of $k_1, \ldots, k_m \geq 1$.

Proof. First rewrite

$$\prod_{i=1}^{m}(n - i + 1) = n^m \prod_{i=1}^{m-1}\left(1 - \frac{i}{n}\right).$$

Then

$$\begin{aligned}
\phi_n^m(k_1, k_2, \ldots, k_m) &= \prod_{i=1}^{m-1}\left(1 - \frac{i}{n}\right)\mathbb{E}\left[\prod_{i=1}^{m}\left(\frac{n\mu}{\sum_{j=i}^{m} k_j + \Xi_{n-m}}\right)\right] \\
&= \mathbb{E}\left[\exp\left(\sum_{i=1}^{m-1}\log\left(1 - \frac{i}{n}\right) - \sum_{i=1}^{m}\log\left(\frac{\Xi_{n-m}}{n\mu} + \frac{1}{n\mu}\sum_{j=i}^{m}k_j\right)\right)\right].
\end{aligned}$$

Now note that for any $x \in (-1, \infty)$, we have $\log(1 + x) \leq x$. We also have $\log(1 - i/n) \geq -i/n - m^2/n^2$ for $1 \leq i \leq m - 1$. So

$$\begin{aligned}
&\phi_n^m(k_1, k_2, \ldots, k_m) \\
&\geq \mathbb{E}\left[\exp\left(-\sum_{i=1}^{m-1}\frac{i}{n} - \frac{m^3}{n^2} - m\left[\frac{\Xi_{n-m}}{n\mu} - 1\right] - \frac{1}{n\mu}\sum_{i=1}^{m}\sum_{j=i}^{m}k_j\right)\right] \\
&= \exp\left(-\frac{m(m-1)}{2n} - \frac{m^3}{n^2} + m + \frac{1}{n\mu}\sum_{i=1}^{m}\Big(s(i) - s(m) - 2(m - i + 1)\Big)\right) \\
&\quad\times \mathbb{E}\left[\exp\left(-\frac{m}{n\mu}\Xi_{n-m}\right)\right] \\
&= \exp\left(-\frac{m(m-1)}{2n} - \frac{m^3}{n^2} + m + \frac{1}{n\mu}\sum_{i=0}^{m}(s(i) - s(m)) - \frac{m(m+1)}{n\mu}\right) \\
&\quad\times \mathbb{E}\left[\exp\left(-\frac{m}{n\mu}\Xi_{n-m}\right)\right] \\
&= \exp\left(\frac{1}{n\mu}\sum_{i=0}^{m}(s(i) - s(m))\right)\exp\left(m - \frac{(2+\mu)}{2\mu}\frac{m^2}{n}\right)\left[\mathcal{L}\left(\frac{m}{n\mu}\right)\right]^{n-m} \\
&\quad\times \exp\left(\frac{(\mu - 2)m}{2\mu n} - \frac{m^3}{n^2}\right).
\end{aligned}$$

We have $m^3/n^2 = O(n^{\frac{\alpha-2}{\alpha+1}}) = o(1)$ and so the final exponential tends to 1 as $n \to \infty$. The desired result then follows from Lemma 2.4.6. □

Lemma 2.4.8. *Let $(X_n)_{n\geq 1}$, $(Y_n)_{n\geq 1}$ be two sequences of non-negative random variables such that $X_n \geq Y_n$ and $\mathbb{E}[X_n] = 1$ for all n. Suppose that X is another non-negative random variable such that $\mathbb{E}[X] = 1$ and $Y_n \xrightarrow{d} X$. Then $X_n \xrightarrow{d} X$ and $(X_n)_{n\geq 1}$ is uniformly integrable.*

Proof. We first prove that $X_n \xrightarrow{d} X$, using Portmanteau's theorem. Let f be a bounded and uniformly continuous function, we have $\mathbb{E}[f(Y_n)] \to \mathbb{E}[f(X)]$. It is enough to show that $\mathbb{E}[f(Y_n + Z_n)] - \mathbb{E}[f(Y_n)] \to 0$, where $Z_n := X_n - Y_n$. Z_n is a non-negative random variable such that $\mathbb{E}[Z_n] \to 0$, hence Z_n converges to 0 in probability. Fix $\varepsilon > 0$, and let $\delta > 0$ be such that $|f(x_1) - f(x_2)| \leq \varepsilon$ for all $x_1, x_2 \in \mathbb{R}$ satisfying $|x_1 - x_2| \leq \delta$. For n large enough, $\mathbb{P}(|Z_n| \geq \delta) \leq \delta$. Thus,

$$|\mathbb{E}[f(Y_n + Z_n)] - \mathbb{E}[f(Y_n)]| \leq \varepsilon + \delta \|f\|_\infty.$$

Since ε and δ can be chosen arbitrarily small, this yields $X_n \xrightarrow{d} X$.
Now, we prove that $X_n \to X$ in L^1, which implies the uniform integrability. By Skorokhod's representation theorem, we may work on a probability space where $X_n \to X$ almost surely.

Consider $(X - X_n)_+$. For any given $\epsilon > 0$, we may choose K such that $\mathbb{E}\left[X \mathbb{1}_{\{X \geq K\}}\right] < \epsilon$ (since $\mathbb{E}[X] < \infty$). So $\mathbb{E}\left[(X - X_n)_+ \mathbb{1}_{\{X \geq K\}}\right] < \epsilon$. But also $\mathbb{E}\left[(X - X_n)_+ \mathbb{1}_{\{X < K\}}\right] \to 0$ as $n \to \infty$ because it is less than $\delta + K\mathbb{P}((X - X_n)_+ > \delta)$ for any $\delta > 0$. So $\mathbb{E}[(X - X_n)_+] \to 0$.

Since $\mathbb{E}[X] = \mathbb{E}[X_n]$, we also have $\mathbb{E}[(X_n - X)_+] = \mathbb{E}[(X - X_n)_+] \to 0$. So finally $\mathbb{E}[|X_n - X|] \to 0$. □

Proof of Proposition 2.4.3. Recall that $S(k) = \sum_{i=1}^{k}(Z_i - 2)$. By Lemma 2.4.7, for $m = \lfloor t n^{\frac{\alpha}{\alpha+1}} \rfloor$,

$$\Phi(n, m) \geq \underline{\Phi}(n, m) := \exp\left(\frac{1}{n\mu}\sum_{i=0}^{m}(S(i) - S(m)) - \frac{C_\alpha m^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1} n^\alpha}\right)(1 + o(1)).$$

By (2.15) and Lemma 2.4.4, we have

$$\frac{1}{n\mu}\sum_{i=0}^{m}(S(i) - S(m)) \xrightarrow{d} \frac{1}{\mu}\int_0^t (L_s - L_t)ds.$$

Hence, by the continuous mapping theorem,

$$\begin{aligned}\underline{\Phi}(n, \lfloor t n^{\frac{\alpha}{\alpha+1}} \rfloor) \xrightarrow{d} &\exp\left(\frac{1}{\mu}\int_0^t (L_s - L_t)ds - \frac{C_\alpha t^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right) \\ &= \exp\left(-\frac{1}{\mu}\int_0^t s dL_s - \frac{C_\alpha t^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right),\end{aligned}$$

and the right-hand side is, of course, $\Phi(t)$, which has mean 1. We also have $\mathbb{E}\left[\Phi(n, \lfloor t n^{\frac{\alpha}{\alpha+1}} \rfloor)\right] = 1$ for all n. So by Lemma 2.4.8, we must have

$$\Phi(n, \lfloor t n^{\frac{\alpha}{\alpha+1}} \rfloor) \xrightarrow{d} \Phi(t)$$

as $n \to \infty$, as well as the claimed uniform integrability. □

We are now ready to prove Theorem 2.4.1.

Proof of Theorem 2.4.1. It is sufficient to show that for any $t \geq 0$ and any bounded continuous test-function $f : \mathbb{D}([0,t], \mathbb{R})^2 \to \mathbb{R}$,

$$\mathbb{E}\left[f\left(n^{-\frac{1}{\alpha+1}}\widetilde{S}^n(\lfloor n^{\frac{\alpha}{\alpha+1}}u\rfloor), n^{-\frac{\alpha-1}{\alpha+1}}\widetilde{G}^n(\lfloor n^{\frac{\alpha}{\alpha+1}}u\rfloor), 0 \leq u \leq t\right)\right] \to \mathbb{E}\left[f(\widetilde{L}_u, \widetilde{H}_u, 0 \leq u \leq t)\right],$$

as $n \to \infty$. Let us write $\overline{S}^n(u) = n^{-\frac{1}{\alpha+1}}S(\lfloor n^{\frac{\alpha}{\alpha+1}}u\rfloor)$ and, similarly, $\overline{G}^n(u) = n^{-\frac{\alpha-1}{\alpha+1}}G(\lfloor n^{\frac{\alpha}{\alpha+1}}u\rfloor)$. Then, by changing measure, we wish to show that for any $t \geq 0$ and any bounded continuous test-function $f : \mathbb{D}([0,t], \mathbb{R})^2 \to \mathbb{R}$,

$$\mathbb{E}\left[\Phi(n, \lfloor tn^{\frac{\alpha}{\alpha+1}}\rfloor)f(\overline{S}^n(u), \overline{G}^n(u), 0 \leq u \leq t)\right] \to \mathbb{E}\left[\Phi(t)f(L_u, H_u, 0 \leq u \leq t)\right],$$

as $n \to \infty$. From the proof of Proposition 2.4.3, we have that

$$\mathbb{E}\left[\left|\Phi(n, \lfloor tn^{\frac{\alpha}{\alpha+1}}\rfloor) - \underline{\Phi}(n, \lfloor tn^{\frac{\alpha}{\alpha+1}}\rfloor)\right|\right] \to 0$$

as $n \to \infty$, and so it will suffice to show that

$$\mathbb{E}\left[\underline{\Phi}(n, \lfloor tn^{\frac{\alpha}{\alpha+1}}\rfloor)f(\overline{S}^n(u), \overline{G}^n(u), 0 \leq u \leq t)\right] \to \mathbb{E}\left[\Phi(t)f(L_u, H_u, 0 \leq u \leq t)\right].$$

But

$$\underline{\Phi}(n, \lfloor tn^{\frac{\alpha}{\alpha+1}}\rfloor) = \exp\left(\frac{1}{\mu}\int_0^t (\overline{S}^n(u) - \overline{S}^n(t))du - \frac{C_\alpha \lfloor tn^{\frac{\alpha}{\alpha+1}}\rfloor^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}n^\alpha}\right).$$

In particular, for a path $x \in \mathbb{D}([0,t], \mathbb{R})$, let

$$\Theta(x,t) = \exp\left(\frac{1}{\mu}\int_0^t (x(u) - x(t))du - \frac{C_\alpha t^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right)$$

and observe that Θ is a continuous functional of its first argument. Then we have

$$\mathbb{E}\left[\left|\underline{\Phi}(n, \lfloor tn^{\frac{\alpha}{\alpha+1}}\rfloor) - \Theta(\overline{S}^n, t)\right|\right] \to 0.$$

So it suffices to show that

$$\mathbb{E}\left[\Theta(\overline{S}^n, t)f(\overline{S}^n(u), \overline{G}^n(u), 0 \leq u \leq t)\right] \to \mathbb{E}\left[\Theta(L, t)f(L_u, H_u, 0 \leq u \leq t)\right].$$

But this now follows from Theorem 2.2.4 and uniform integrability. □

2.5 The configuration multigraph

The processes $\widetilde{S}^n$ and $\widetilde{G}^n$ encode a forest of trees where the numbers of children of the vertices, visited in depth-first order, are $\hat{D}_i^n - 1$, $i \geq 1$. Let us write $\widetilde{\mathbf{F}}_n(\nu)$ for this forest. In this section, we wish to encode similarly the multigraph $\mathbf{M}_n(\nu)$. Let us first describe the organisation of this section.

In Section 2.5.1, we simultaneously generate and explore $\mathbf{M}_n(\nu)$ using the depth-first approach outlined in the Introduction: we view each connected component of the graph as a spanning tree explored in a depth-first manner plus some additional edges, creating cycles, that we call *back-edges*. In Section 2.5.2, we prove that the exploration process is close enough to $\widetilde{S}^n$ in order to have the same scaling limit, and add the joint convergence of the locations of the back-edges in Section 2.5.3 (Theorem 2.5.5). In Section 2.5.4, we split the multigraph into its components by showing that the (rescaled) ordered sequence of component sizes converges to the sequence of ordered excursion lengths of the continuous process R (Proposition 2.5.6). We improve this result in Section 2.5.5 by adding in the locations of the back-edges under each excursion (Proposition 2.5.12). In Section 2.5.6, we study the height process and show in Proposition 2.5.16 the joint convergence of the height process excursions and the locations of the back-edges. This finally allows us to prove Theorem 2.1.1 in Section 2.5.7.

2.5.1 Exploration of the multigraph

We work conditionally on the sequence $(\hat{D}_1^n, \hat{D}_2^n, \dots, \hat{D}_n^n)$. Let us declare that the vertex of degree $\hat{D}_i^n$ is called v_i. This means that we have already determined the (size-biased by degree) order in which we will observe new vertices. We will couple $\widetilde{\mathbf{F}}_n(\nu)$ and $\mathbf{M}_n(\nu)$ by using the same ordering on the new vertices we explore.

Recall that we start from vertex v_1 with degree $\hat{D}_1^n$. We maintain a *stack*, namely an ordered list of half-edges which we have seen but not yet explored (remember that the half-edges come with an arbitrary labelling for this purpose). We put the $\hat{D}_1^n$ half-edges of v_1 onto this stack, in increasing order of label, so that the lowest labelled half-edge is on top of the stack. At a subsequent step, suppose we have already seen the vertices $v_1, v_2, \dots, v_k$. If the stack is non-empty, take the top half-edge and sample its pair. This lies on the stack with probability proportional to the height of the stack minus 1 or belongs to v_{k+1} with probability proportional to $\sum_{i=k+1}^n \hat{D}_i^n$. In the first case we simply remove both half-edges from the stack. In the second, we remove the half-edge at the top of the stack (which has just been paired) and replace it by the remaining half-edges (if any) of v_{k+1}. If the stack is empty, we start a new component at v_{k+1}.

Let us now describe the forest $\mathbf{F}_n(\nu)$ from which we will recover $\mathbf{M}_n(\nu)$. Whenever there is a back-edge in $\mathbf{M}_n(\nu)$, say from vertex v_k to vertex v_i with $i \le k$, remove the back-edge and replace it by two edges, one from v_k to a new leaf and the other from v_i to a new leaf. To recover $\mathbf{M}_n(\nu)$ it is then sufficient to remove the edges to the new leaves and put in a new edge from v_i to v_k.

Our aim is to encode the forest $\mathbf{F}_n(\nu)$, firstly via its depth-first walk and then by its height process. We will simultaneously keep track of marks which tell us which vertices we should identify in order to recover the multigraph.

Our first observation is that the vertex-sets of *pairs* of components in $\widetilde{\mathbf{F}}_n(\nu)$ correspond precisely to the vertex-sets of subcollections of components in $\mathbf{M}_n(\nu)$. (This is illustrated in Figure 2.1 to which the reader is referred in the following argument.) More precisely, without loss of generality, let the pair of components of $\widetilde{\mathbf{F}}_n(\nu)$ be the first two, on vertices $v_1, v_2, \ldots, v_{m_1}$ and $v_{m_1+1}, v_{m_1+2}, \ldots, v_{m_1+m_2}$. Suppose that the same vertices in $\mathbf{M}_n(\nu)$ are adjacent to b back-edges. Now vertices v_1 and v_{m_1+1} each possess one more half-edge in $\mathbf{M}_n(\nu)$ than they do in $\widetilde{\mathbf{F}}_n(\nu)$ (since in $\widetilde{\mathbf{F}}_n(\nu)$, if v_i is the first vertex of a component, it has $\hat{D}_i - 1$ edges and not $\hat{D}_i$). In particular, adding an edge between v_1 and v_{m_1+1} clearly produces a tree with $m_1 + m_2$ vertices and $\frac{1}{2}\sum_{i=1}^{m_1+m_2} \hat{D}_i^n = m_1 + m_2 - 1$ edges. We now "rewire" this tree to obtain the relevant components of $\mathbf{M}_n(\nu)$. The effect of adding a back-edge is to shunt all of the subsequent subtrees along in the depth-first order. (See Figure 2.1.) The overall effect is that each back-edge causes a new component to come into existence. Each time we observe a back-edge, it occupies two half-edges, so there are two subtrees which get pushed out of the component. The earlier of these subtrees in the depth-first order becomes the basis for the next component of $\mathbf{M}_n(\nu)$. The root of this component has one more child than it had in the original tree. This allows the absorption of the second subtree, whose root gets attached by its free half-edge to the root of the component. Subsequent back-edges similarly each generate one new component. Following this through, we see that we end up with $b+1$ components of $\mathbf{M}_n(\nu)$. (For the purposes of intuition, note that because the vast majority of vertices lie in components of size $o(n^{\alpha/(\alpha+1)})$, with high probability at most one of them will be of size $\Theta(n^{\alpha/(\alpha+1)})$ and thus show up in the limit. So, at least heuristically, this rewiring process cannot affect what we see in the limit.)

It is clear that the effects of adding back-edges are relatively local and so it is at least intuitively clear that the depth-first walk of the forest $\mathbf{F}_n(\nu)$ should be similar to that of $\widetilde{\mathbf{F}}_n(\nu)$, as long as there are not too many back-edges. Let X^n denote the depth-first walk of $\mathbf{F}_n(\nu)$. We will now describe how to construct X^n from $\widetilde{S}^n$, and also how to keep track of the back-edges. We will write $R^n(k)$ for the number of half-edges on the stack at step k. We will let N^n count the occurrences of back-edges, and U^n the positions of their targets on the stack. We will write $\mathcal{M}^n(k)$ for a set of marks (in $\mathbb{N}$) at step k, indicating back-edges which have not yet been closed, and $\tau_n(k)$ for the number of vertices already seen at step k (note that we see a new vertex if and only if the current step does not involve a back-edge). Finally, let $C^n(k)$ be the number of components of $\mathbf{F}_n(\nu)$ we have fully explored by time k. So we will have that for all $k \geq 1$, $C^n(k) = -\min_{0 \leq \ell \leq k} X^n(\ell)$, and that $R^n(k) = X^n(k) + C^n(k)$.

We start from $X^n(0) = N^n(0) = 0$, $\mathcal{M}^n(0) = \emptyset$ and $\tau_n(0) = 0$. For $k \geq 0$, we might encounter the following three situations.

- **New component.**
  If $X^n(k) = \min_{0 \leq i \leq k-1} X^n(i) - 1$ or $k = 0$, let $\tau_n(k+1) = \tau_n(k) + 1$, $N^n(k+1) = N^n(k)$, $\mathcal{M}^n(k+1) = \mathcal{M}^n(k)$ and

$$X^n(k+1) = X^n(k) + \widetilde{S}^n(\tau_n(k) + 1) - \widetilde{S}^n(\tau_n(k)) + 1.$$

- **Start a back-edge or not.**

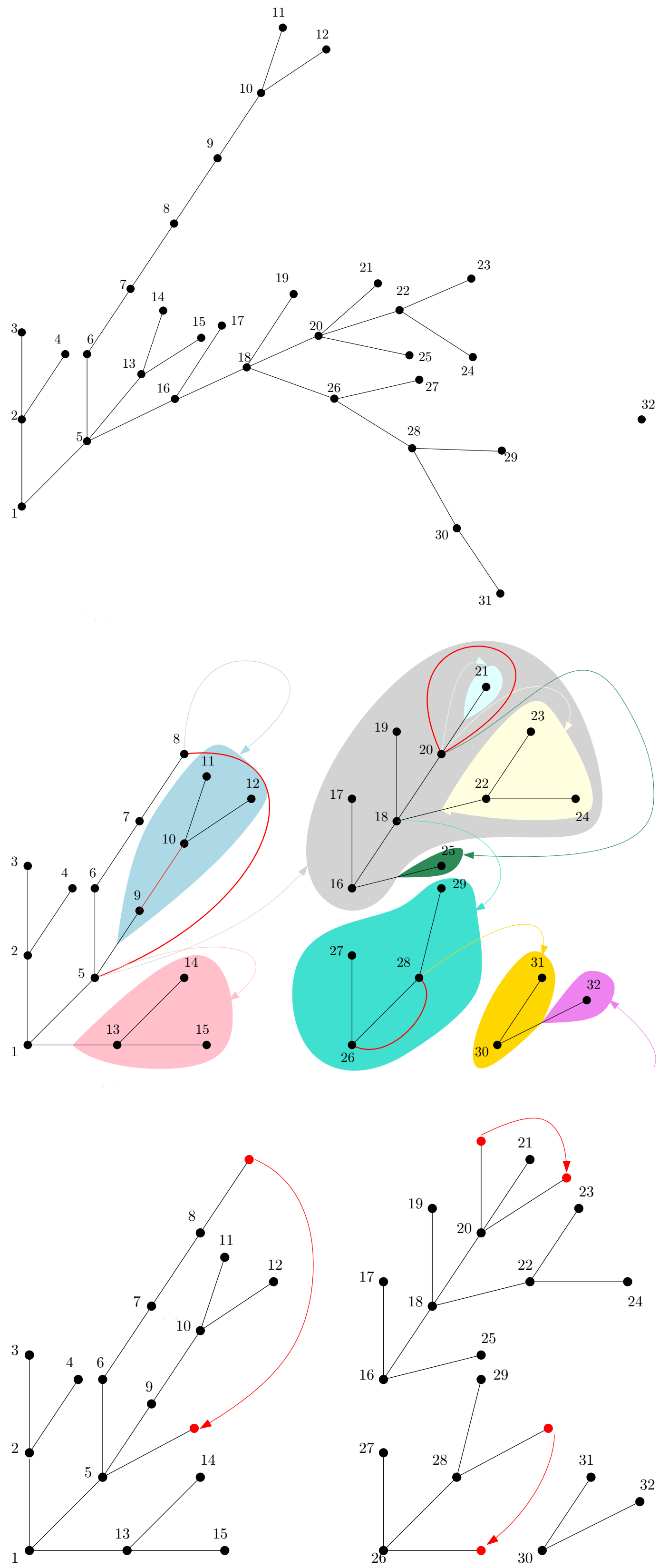

Figure 2.1: For simplicity, the labels given are those corresponding to the depth-first order. Top: the first two components of the forest $\widetilde{\mathbf{F}}_n(\nu)$. Middle: the first four components of $\mathbf{M}_n(\nu)$, on the same vertices. Three back-edges are marked in red. The subtree surgery required to get from $\widetilde{\mathbf{F}}_n(\nu)$ to $\mathbf{M}_n(\nu)$ is indicated. The back-edge from 5 to 8 moves the subtree from 9 to the next available half-edge, also belonging to 5. This shifts further the subtrees from 13 and 16: 13 is connected to 1, which has one more edge in $\mathbf{M}_n(\nu)$ than in $\widetilde{\mathbf{F}}_n(\nu)$ ($\hat{D}_1$ instead of $\hat{D}_1 - 1$), as any first vertex of a connected component of $\mathbf{M}_n(\nu)$. 16 starts a new component. Bottom:

If $X^n(k) > \min_{0\le i\le k-1} X^n(i) - 1$ and $X^n(k) \notin \mathcal{M}^n(k)$,

– With probability

$$\frac{X^n(k) - \min_{0\le i\le k} X^n(k) - |\mathcal{M}^n(k)|}{X^n(k) - \min_{0\le i\le k} X^n(k) - |\mathcal{M}^n(k)| + \sum_{j=\tau_n(k)+1}^n \hat{D}_j^n},$$

let $\tau_n(k+1) = \tau_n(k)$ and $X^n(k+1) = X^n(k) - 1$. Let $N^n(k+1) = N^n(k) + 1$, sample $U^n(k+1)$ uniformly from

$$\left\{\min_{0\le i\le k} X^n(i), \ldots, X^n(k) - 1\right\} \setminus \mathcal{M}^n(k),$$

and let $\mathcal{M}^n(k+1) = \mathcal{M}^n(k) \cup \{U^n(k+1)\}$.

– With probability

$$\frac{\sum_{j=\tau_n(k)+1}^n \hat{D}_j^n}{X^n(k) - \min_{0\le i\le k} X^n(k) - |\mathcal{M}^n(k)| + \sum_{j=\tau_n(k)+1}^n \hat{D}_j^n}$$

let $\tau_n(k+1) = \tau_n(k) + 1$,

$$X^n(k+1) = X^n(k) + \widetilde{S}^n(\tau_n(k) + 1) - \widetilde{S}^n(\tau_n(k)),$$

$N^n(k+1) = N^n(k)$ and $\mathcal{M}^n(k+1) = \mathcal{M}^n(k)$.

- **Close a back-edge?**
  If $X^n(k) > \min_{0\le i\le k-1} X^n(i) - 1$ and $X^n(k) \in \mathcal{M}^n(k)$ then let $\tau_n(k+1) = \tau_n(k)$, $X^n(k+1) = X^n(k) - 1$, $N^n(k+1) = N^n(k)$, and $\mathcal{M}^n(k+1) = \mathcal{M}^n(k) \setminus \{X^n(k)\}$.

It is straightforward to check that this is the depth-first walk of the forest $\mathbf{F}^n(\nu)$. We observe that for $k \ge 1$ we have

$$X^n(k) + \min_{0\le i\le k} X^n(i) = \widetilde{S}^n(\tau_n(k)) + 1 - N^n(k) - \#\{i \le k : |\mathcal{M}^n(i)| < |\mathcal{M}^n(i-1)|\}.$$

Hence, $X^n(k) + \min_{0\le i\le k} X^n(i)$ is the number of half-edges seen but not yet paired or reserved for back-edges, in the currently explored connected component (the $(j+1)$-th component if $-\min_{0\le i\le k} X^n(i) = j$). Indeed,

$$\widetilde{S}^n(\tau_n(k)) + 1 = (\hat{D}_1 - 1) - 1 + \hat{D}_2 - 2 + \ldots + \hat{D}_{\tau_n(k)-1} - 2 + \hat{D}_{\tau_n(k)} - 1$$

is the number of half-edges seen and not paired in the spanning forest of the components explored so far. $N^n(k) + \#\{i \le k : |\mathcal{M}^n(i)| < |\mathcal{M}^n(i-1)|\}$ is the number of half-edges that have been paired in or reserved for back-edges. Note that $|\mathcal{M}^n(i)|$ increases (resp. decreases) by 1 exactly when a back-edge is initiated (resp. closed). $N^n(k)$ counts the number of back-edges started.

In particular,

$$\widetilde{S}^n(\tau_n(k)) - 2N^n(k) \le X^n(k) + \min_{0\le i\le k} X^n(i) - 1 \le \widetilde{S}^n(\tau_n(k)). \tag{2.17}$$

2.5.2 Convergence of the depth-first walk and marks

Let us first prove a bound on the number $N^n(k)$ of back-edges which have occurred by step k.

Lemma 2.5.1. *For every $t > 0$, the sequences of random variables $\left(N^n(\lfloor tn^{\alpha/(\alpha+1)}\rfloor)\right)_{n\geq 1}$ and $\left(\sup_{0\leq k\leq \lfloor tn^{\alpha/(\alpha+1)}\rfloor} |\tau_n(k)-k|\right)_{n\geq 1}$ are tight.*

Proof. Fix $t > 0$. We observe that $k - 2N^n(k) \leq \tau_n(k) \leq k$ so that it is enough to prove that $\left(N^n(\lfloor tn^{\alpha/(\alpha+1)}\rfloor)\right)_{n\geq 1}$ is tight. At time i, the number of half-edges on the stack is $R^n(i)$, and the total number of unpaired half-edges is $\sum_{j=\tau_n(i)+1}^{n} \hat{D}_j^n$, so that the probability to start a back-edge is

$$\frac{R^n(i)}{\sum_{j=\tau_n(i)+1}^{n} \hat{D}_j^n}$$

Note that if the component we are exploring at step i began at step j, then $R^n(i) \leq 2 + \sum_{k=\tau_n(j)}^{\tau_n(i)} (\hat{D}_k^n - 2)$, since at most $\hat{D}_{\tau_n(j)}^n + \ldots + \hat{D}_{\tau_n(i)}^n$ half-edges have been seen, and at least $2(\tau_n(i) - \tau_n(j))$ of them have been used to connect the first $\tau_n(i) - \tau_n(j) + 1$ vertices of the component. Hence,

$$R^n(i) \leq 2 + \max_{0\leq j\leq i} \widetilde{S}^n(\tau_n(j)) - \min_{0\leq j\leq i} \widetilde{S}^n(\tau_n(j)) \leq 2 + \max_{0\leq j\leq i} \widetilde{S}^n(j) - \min_{0\leq j\leq i} \widetilde{S}^n(j),$$

since $\tau_n(\{1,\ldots,i\}) \subseteq \{1,\ldots,i\}$.

As underlined in the beginning of Section 2.5.1, we can realize $\mathbf{M}_n(\nu)$ by first drawing the sequence $(\hat{D}_i^n)_{1\leq i\leq n}$, and then proceeding to the pairings of the half-edges through the exploration. Conditionally on $(\hat{D}_i^n)_{1\leq i\leq n}$, for every $i \geq 1$ and whatever happened in the first $i-1$ steps of the exploration, the probability to create a back-edge at time i is at most

$$\frac{2 + \max_{0\leq j\leq \lfloor tn^{\alpha/(\alpha+1)}\rfloor} \widetilde{S}^n(j) - \min_{0\leq j\leq \lfloor tn^{\alpha/(\alpha+1)}\rfloor} \widetilde{S}^n(j)}{\sum_{j=\tau_n(i)+1}^{n} \hat{D}_j^n}.$$

Note also that $\tau_n(i) + 1 \leq \lfloor tn^{\alpha/(\alpha+1)}\rfloor + 1$ for every $i \leq \lfloor tn^{\alpha/(\alpha+1)}\rfloor$. Therefore, conditionally on $(\hat{D}_i^n)_{1\leq i\leq n}$, the random variable $N^n(\lfloor tn^{\alpha/(\alpha+1)}\rfloor)$ is stochastically dominated by a Binomial random variable with parameters $\lfloor tn^{\alpha/(\alpha+1)}\rfloor$ and

$$\frac{2 + \max_{0\leq j\leq \lfloor tn^{\alpha/(\alpha+1)}\rfloor} \widetilde{S}^n(j) - \min_{0\leq j\leq \lfloor tn^{\alpha/(\alpha+1)}\rfloor} \widetilde{S}^n(j)}{\sum_{j=\lfloor tn^{\alpha/(\alpha+1)}\rfloor+1}^{n} \hat{D}_j^n}.$$

For $K > 0$, define

$$\mathcal{E}_1 = \left\{2 + \max_{1\leq i\leq \lfloor tn^{\alpha/(\alpha+1)}\rfloor} \widetilde{S}^n(i) - \min_{0\leq i\leq \lfloor tn^{\alpha/(\alpha+1)}\rfloor} \widetilde{S}^n(i) \leq Kn^{1/(\alpha+1)}\right\}$$

and

$$\mathcal{E}_2 = \left\{\sum_{j=\lfloor tn^{\alpha/(\alpha+1)}\rfloor+1}^{n} \hat{D}_j^n \geq n\right\}.$$

Fix $\epsilon > 0$. Theorem 2.4.1, Lemma 2.6.5 and the fact that $\mu > 1$ imply that there exists $K > 0$ such that for n large enough,

$$\mathbb{P}\left(\mathcal{E}_1 \cap \mathcal{E}_2\right) \geq 1 - \epsilon.$$

On the event $\mathcal{E}_1 \cap \mathcal{E}_2$, we have

$$\frac{2 + \max_{0 \leq j \leq \lfloor tn^{\alpha/(\alpha+1)} \rfloor} \widetilde{S}^n(j) - \min_{0 \leq j \leq \lfloor tn^{\alpha/(\alpha+1)} \rfloor} \widetilde{S}^n(j)}{\sum_{j=\lfloor tn^{\alpha/(\alpha+1)} \rfloor + 1}^{n} \hat{D}_j^n} \leq \frac{K}{n^{\alpha/(\alpha+1)}}.$$

Let $Y \sim \mathrm{Bin}(\lfloor tn^{\alpha/(\alpha+1)} \rfloor, 2K/n^{\alpha/(\alpha+1)})$. Then there exists $K' > 0$ such that for n large enough,

$$\mathbb{P}\left(N^n(\lfloor tn^{\alpha/(\alpha+1)} \rfloor) \geq K'\right) \leq \mathbb{P}\left(Y \geq K'\right) + \mathbb{P}\left((\mathcal{E}_1 \cap \mathcal{E}_2)^c\right) \leq 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, the result follows. □

In particular, the steps on which back-edges occur are negligible on the timescale in which we are interested. Write $\widetilde{I}_t = \inf_{0 \leq s \leq t} \widetilde{L}_s$ for $t \geq 0$ and recall that

$$R_t = \widetilde{L}_t - \widetilde{I}_t.$$

Proposition 2.5.2. *As $n \to \infty$,*

$$\left(n^{-1/(\alpha+1)} \widetilde{S}^n(\lfloor tn^{\alpha/(\alpha+1)} \rfloor), n^{-1/(\alpha+1)} R^n(\lfloor tn^{\alpha/(\alpha+1)} \rfloor), n^{-1/(\alpha+1)} C^n(\lfloor tn^{\alpha/(\alpha+1)} \rfloor), t \geq 0\right)$$
$$\xrightarrow{d} \left(\widetilde{L}_t, R_t, -\tfrac{1}{2}\widetilde{I}_t, t \geq 0\right),$$

in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})^3$.

Proof. Let $k \geq 1$, and let j be such that $X^n(j) = \min_{0 \leq \ell \leq k} X^n(\ell)$. Then $X^n(j) + \min_{0 \leq \ell \leq j} X^n(\ell) = 2X^n(j)$, and by (2.17), we have

$$\begin{aligned} 2X^n(j) \geq& 1 + \min_{0 \leq \ell \leq k} (\widetilde{S}^n(\tau_n(\ell)) - 2N^n(\ell)) \\ \geq& 1 + \min_{0 \leq \ell \leq k} \widetilde{S}^n(\tau_n(\ell)) - 2N^n(k), \end{aligned}$$

since $(N^n(i))_{i \geq 0}$ is non-increasing. Thus, by (2.17), we have

$$1 + \min_{0 \leq \ell \leq k} \widetilde{S}^n(\tau_n(\ell)) - 2N^n(k) \leq 2 \min_{0 \leq \ell \leq k} X^n(\ell) \leq 1 + \min_{0 \leq \ell \leq k} \widetilde{S}^n(\tau_n(\ell)).$$

By Theorem 2.4.1 and the continuous mapping theorem,

$$\left(n^{-1/(\alpha+1)} \min_{0 \leq \ell \leq \lfloor sn^{\alpha/(\alpha+1)} \rfloor} \widetilde{S}^n(\ell), s \geq 0\right) \xrightarrow{d} (\widetilde{I}_s, s \geq 0)$$

and combining this with Lemma 2.5.1 and recalling that $C^n(k) = -\min_{0 \leq \ell \leq k} X^n(\ell)$ yields

$$\left(n^{-1/(\alpha+1)} C^n(\lfloor sn^{\alpha/(\alpha+1)} \rfloor), s \geq 0\right) \xrightarrow{d} (-\tfrac{1}{2}\widetilde{I}_s, s \geq 0)$$

Another application of (2.17) gives

$$1 + \widetilde{S}^n(\tau_n(k)) - 2N^n(k) - 2\min_{0 \le i \le k} X^n(i) \le R^n(k) \le 1 + \widetilde{S}^n(\tau_n(k)) - 2\min_{0 \le i \le k} X^n(i)$$

and since

$$(n^{-1/(\alpha+1)}\widetilde{S}^n(\tau_n(\lfloor tn^{\alpha/(\alpha+1)}\rfloor)), t \ge 0) \xrightarrow{d} (\widetilde{L}_t, t \ge 0)$$

we have

$$\left(n^{-1/(\alpha+1)}R^n(\lfloor tn^{\alpha/(\alpha+1)}\rfloor), t \ge 0\right) \xrightarrow{d} (R_t, t \ge 0),$$

jointly with the convergence of the minimum. □

Thus the exploration of $\widetilde{\mathbf{F}}_n(\nu)$ sees approximately twice as many components as that of $\mathbf{F}_n(\nu)$ but the limiting reflected process is the same for both. In particular, asymptotically the two processes have the same longest excursions. This fact will play an important role in the sequel.

2.5.3 Back-edges

We will now show that the parts of the multigraph we observe up until well beyond the timescale in which we are interested are, with high probability, simple. To this end, let $A^n(k)$ be the number of loops and edges created parallel to an existing edge, up until step k of the depth-first exploration of $\mathbf{F}_n(\nu)$. Call these *anomalous* edges.

Proposition 2.5.3. *Suppose $\frac{\alpha}{\alpha+1} < \beta < \frac{\alpha}{2}$. Then we have*

$$\mathbb{P}\left(A^n(\lfloor n^\beta \rfloor) > 0\right) \to 0$$

as $n \to \infty$.

Proof. We adapt the proof of Lemma 7.1 of Joseph [91] (which applies in the finite third moment setting). Self-loops are obviously associated with a unique vertex. We associate extra edges created parallel to an existing edge with their vertex which is discovered first in the depth-first exploration. Consider a particular vertex of degree d in the exploration before time $\lfloor n^\beta \rfloor$. Its kth half-edge (in the order that we process them) creates a self-loop with probability bounded above by

$$\frac{d-k}{\sum_{i=\lfloor n^\beta \rfloor+1}^n \hat{D}_i^n}.$$

It creates a multiple edge with probability at most

$$\frac{k-1}{\sum_{i=\lfloor n^\beta \rfloor+1}^n \hat{D}_i^n}.$$

This vertex therefore possesses an anomalous edge with probability bounded above by

$$\frac{d(d-1)}{\sum_{i=\lfloor n^\beta \rfloor+1}^n \hat{D}_i^n}.$$

Hence, by the conditional version of Markov's inequality,

$$\mathbb{P}\left(A^n(\lfloor n^\beta \rfloor) > 0 \;\middle|\; \hat{D}_1^n, \hat{D}_2^n, \ldots, \hat{D}_n^n\right) \le \left(\frac{\sum_{i=1}^{\lfloor n^\beta \rfloor} (\hat{D}_i^n)^2}{\sum_{i=\lfloor n^\beta \rfloor+1}^{n} \hat{D}_i^n}\right) \wedge 1.$$

But $\sum_{i=\lfloor n^\beta \rfloor+1}^{n} \hat{D}_i^n = \sum_{i=1}^{n} \hat{D}_i^n - \sum_{i=1}^{\lfloor n^\beta \rfloor} \hat{D}_i^n \ge \sum_{i=1}^{n} \hat{D}_i^n - \sum_{i=1}^{\lfloor n^\beta \rfloor} (\hat{D}_i^n)^2$. By Lemmas 2.6.4 and 2.6.5 and the bounded convergence theorem, we obtain that

$$\mathbb{P}\left(A^n(\lfloor n^\beta \rfloor) > 0\right) \to 0$$

as $n \to \infty$. □

Let $\rho(n) = \inf\{k \ge 0 : A^n(k) > 0\}$ and note that the event that $\mathbf{M}_n(\nu)$ is simple is equal to $\{\rho(n) = \infty\}$. The last proposition shows that we observe any anomalous edges long after the timescale in which we explore the largest components of the graphs. This allows us to conclude that all of the results we prove using only the timescale $n^{\alpha/(\alpha+1)}$ for the multigraph are also true conditionally on $\{\rho(n) = \infty\}$. In this way, we may give a proof of Conjecture 8.6 of Joseph [91]. (See also Theorem 3 of [62].)

Theorem 2.5.4. *Conditional on* $\{\rho(n) = \infty\}$*, as* $n \to \infty$*,*

$$\left(n^{-1/(\alpha+1)} R^n(\lfloor s n^{\alpha/(\alpha+1)} \rfloor), n^{-1/(\alpha+1)} C^n(\lfloor s n^{\alpha/(\alpha+1)} \rfloor, s \ge 0\right) \xrightarrow{d} \left(R_s, -\tfrac{1}{2}\widetilde{I}_s, s \ge 0\right),$$

in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})^2$*.*

Proof. Given Propositions 2.5.2 and 2.5.3, this follows in exactly the manner as Joseph's Theorem 3.2 follows from his Theorem 3.1. □

Henceforth, using exactly the same argument, statements about our processes should be understood to hold either unconditionally or conditionally on the event $\{\rho(n) = \infty\}$.

We now turn to the locations of the back-edges that do occur. Recall that if a back-edge occurs at step k, then $U^n(k)$ is its index in the stack. For steps k such that $N^n(k) - N^n(k-1) = 0$, declare $U^n(k) = \partial$, where ∂ denotes that no mark occurs.

Theorem 2.5.5. *We have*

$$\begin{aligned}\Big(n^{-1/(\alpha+1)} R^n(\lfloor s n^{\frac{\alpha}{\alpha+1}} \rfloor), n^{-1/(\alpha+1)} C^n(\lfloor s n^{\alpha/(\alpha+1)} \rfloor, N^n(\lfloor s n^{\frac{\alpha}{\alpha+1}} \rfloor),&\\ n^{-1/(\alpha+1)} U^n(\lfloor s n^{\frac{\alpha}{\alpha+1}} \rfloor), s \ge 0\Big)&\\ \xrightarrow{d} (R_s, -\tfrac{1}{2}\widetilde{I}_s, N_s, U_s, s \ge 0),&\end{aligned}$$

where $((N_s, U_s), s \ge 0)$ *is a marked Cox process of intensity* R_s/μ *at time* $s \ge 0$*, and the marks are uniform on* $[0, R_s]$ *i.e.*

$$U_s \begin{cases} = \partial & \text{if } N_s - N_{s-} = 0 \\ \sim \mathrm{U}[0, R_s] & \text{if } N_s - N_{s-} = 1. \end{cases}$$

Equivalently, conditionally on $(R_s, s \geq 0)$,

$$\sum_{\substack{0 \leq s \leq t: \\ N_s - N_{s-} = 1}} \delta_{(s, U_s)}$$

is a Poisson point process on $\{(s,x) \in \mathbb{R}_+ \times \mathbb{R}_+ : x \leq R_s\}$ *of constant intensity* $1/\mu$. *Here, the convergence is in the Skorokhod topology for the first three co-ordinates and in the topology of vague convergence for counting measures on* $\mathbb{R}_+^2$ *for the fourth.*

We observe that, in particular, for fixed $t \geq 0$ we have $\sup_{0 \leq s \leq t} R_s < \infty$ a.s. and so $N_t < \infty$ a.s.

Proof. We refine the argument from the proof of Lemma 2.5.1. At step k, conditionally on the sequence $(\hat{D}_i)_{1 \leq i \leq n}$ and on the first $k-1$ steps of the exploration, the probability of seeing a back-edge is

$$\frac{R^n(k) - |\mathcal{M}^n(k)|}{R^n(k) - |\mathcal{M}^n(k)| + \sum_{j=\tau_n(k)+1}^n \hat{D}_j^n},$$

where $|\mathcal{M}^n(k)| \leq N^n(k)$. Now

$$\sum_{i=k+1}^n \hat{D}_i^n \leq R^n(k) - |\mathcal{M}^n(k)| + \sum_{i=\tau^n(k)+1}^n \hat{D}_i^n \leq \sum_{i=1}^n \hat{D}_i^n + R^n(k).$$

But then by Lemma 2.5.1, Proposition 2.5.2, Lemma 2.6.5 (in the Appendix) and Slutsky's lemma, we obtain

$$n^{\alpha/(\alpha+1)} \left(\frac{R^n(\lfloor sn^{\alpha/(\alpha+1)} \rfloor) - |\mathcal{M}^n(\lfloor sn^{\alpha/(\alpha+1)} \rfloor)|}{R^n(\lfloor sn^{\alpha/(\alpha+1)} \rfloor) - |\mathcal{M}^n(\lfloor sn^{\alpha/(\alpha+1)} \rfloor)| + \sum_{j=\tau_n(\lfloor sn^{\alpha/(\alpha+1)} \rfloor)+1}^n \hat{D}_j^n}, 0 \leq s \leq t \right)$$
$$\xrightarrow{d} \frac{1}{\mu}(R_s, 0 \leq s \leq t). \tag{2.18}$$

Let $\mathcal{F}_k^n = \sigma((\hat{D}_1^n, \hat{D}_2^n, \ldots, \hat{D}_n^n), X^n(i), N^n(i), \mathcal{M}^n(i), 0 \leq i \leq k)$. Then $(N^n(k), k \geq 0)$ is a counting process with compensator

$$N_{\text{comp}}^n(k) = \sum_{j=1}^{k-1} \frac{R^n(j) - |\mathcal{M}^n(j)|}{R^n(j) - |\mathcal{M}^n(j)| + \sum_{i=\tau_n(j)+1}^n \hat{D}_i^n} \mathbb{1}_{\{X^n(j) \notin \mathcal{M}^n(j)\}}.$$

Since

$$\sum_{j=1}^{k-1} \frac{R^n(j) - |\mathcal{M}^n(j)|}{R^n(j) - |\mathcal{M}^n(j)| + \sum_{i=\tau_n(j)+1}^n \hat{D}_i^n} \mathbb{1}_{\{X^n(j) \in \mathcal{M}^n(j)\}}$$
$$\leq N^n(k-1) \max_{0 \leq j \leq k-1} \frac{R^n(j) - |\mathcal{M}^n(j)|}{R^n(j) - |\mathcal{M}^n(j)| + \sum_{i=\tau_n(j)+1}^n \hat{D}_i^n}$$

and $n^{-\alpha/(\alpha+1)} N^n(\lfloor tn^{\alpha/(\alpha+1)} \rfloor) \xrightarrow{p} 0$ by Lemma 2.5.1, using (2.18) and the continuous mapping theorem we get that

$$E^n(\lfloor tn^{\alpha/(\alpha+1)} \rfloor) := \sum_{j=0}^{\lfloor tn^{\alpha/(\alpha+1)} \rfloor - 1} \frac{R^n(j) - |\mathcal{M}^n(j)|}{R^n(j) - |\mathcal{M}^n(j)| + \sum_{i=\tau_n(j)+1}^n \hat{D}_i^n} \mathbb{1}_{\{X^n(j) \in \mathcal{M}^n(j)\}} \xrightarrow{p} 0.$$

So by (2.18) and another application of the continuous mapping theorem, we obtain

$$N^n_{\text{comp}}(\lfloor tn^{\alpha/(\alpha+1)}\rfloor) = \sum_{j=1}^{k-1} \frac{R^n(j) - |\mathcal{M}^n(j)|}{R^n(j) - |\mathcal{M}^n(j)| + \sum_{i=\tau_n(j)+1}^n \hat{D}^n_i} - E^n(\lfloor tn^{\alpha/(\alpha+1)}\rfloor)$$
$$\xrightarrow{d} \frac{1}{\mu}\int_0^t R_s ds.$$

Finally, Theorem 14.2.VIII of Daley and Vere-Jones [60] (with $\mathcal{F}^{(n)}_0 = \sigma(\hat{D}^k_i, k, i \geq 1)$ for all $n \geq 1$, recall the construction of $\mathbf{M}_n(\nu)$ in the beginning of Section 2.5.1) yields that

$$\left(n^{-1/(\alpha+1)} R^n(\lfloor sn^{\alpha/(\alpha+1)}\rfloor), N^n(\lfloor sn^{\alpha/(\alpha+1)}\rfloor), 0 \leq s \leq t\right) \xrightarrow{d} (R_s, N_s, 0 \leq s \leq t).$$

The marks are uniform on the vertices of the stack which do not already carry marks, and so it is straightforward to see that they must be uniform in the limit. □

2.5.4 Components of the finite graph

We now turn to the consideration of the individual components of the multigraph. Let $\sigma^n(0) = 0$ and for $k \geq 1$, write

$$\sigma^n(k) = \inf\{j \geq 0 : C^n(j) \geq k\}.$$

This is the time at which we finish exploring the kth component of the forest $\mathbf{F}_n(\nu)$. Let

$$\zeta^n(k) = \sigma^n(k) - \sigma^n(k-1),$$

the corresponding length of the excursion, which is equal to the total number vertices within the component, since precisely one of these is killed at each step. But then $\zeta^n(k)$ is also equal to the number of vertices in the corresponding component of $\mathbf{M}_n(\nu)$, plus twice the number of back-edges. Let

$$\varepsilon^n_k(t) = n^{-1/(\alpha+1)}\left(X^n(\sigma^n(k-1) + \lfloor tn^{\alpha/(\alpha+1)}\rfloor) - X^n(\sigma^n(k-1))\right)$$

for $0 \leq t \leq n^{-\alpha/(\alpha+1)}\zeta^n(k)$ be the kth rescaled excursion of X^n, with length $\zeta(\varepsilon^n_k) = n^{-\alpha/(\alpha+1)}\zeta^n(k)$ and rescaled left endpoint $g^n_k = n^{-\alpha/(\alpha+1)}\sigma^n(k-1)$.

Recall from Section 2.3 the notation $(\widetilde{\varepsilon}_i, i \geq 1)$ for the ordered excursions of R above 0 and $\zeta(\widetilde{\varepsilon}_i)$ for the lifetime of $\widetilde{\varepsilon}_i$. Denote by g_i the left endpoint of $\widetilde{\varepsilon}_i$. Recall also that $\ell^2_\downarrow = \{(x_1, x_2, \ldots) \in \mathbb{R}^\mathbb{N} : x_1 \geq x_2 \geq \ldots \geq 0, \sum_{i\geq 1} x_i^2 < \infty\}$. Let Γ be a countable index set and write $\ell^2_+(\Gamma)$ for the set of non-negative sequences $(x_\gamma : \gamma \in \Gamma)$ such that $\sum_{\gamma\in\Gamma} x_\gamma^2 < \infty$. Write $\text{ord} : \ell^2_+(\Gamma) \to \ell^2_\downarrow$ for the map which puts the elements of $(x_\gamma : \gamma \in \Gamma)$ into decreasing order. For a sequence $(\varepsilon_k, A_k)_{k\geq 1}$ where ε_k is an excursion of length $\zeta(\varepsilon_k)$ and A_k is some other random variable, write

$$\text{ord}\,(\zeta(\varepsilon_k), A_k, k \geq 1)$$

for the same sequence put in decreasing order of $\zeta(\varepsilon_k)$.

This section is devoted to proving the following proposition.

Proposition 2.5.6. *We have* $(\zeta(\widetilde{\varepsilon}_i), i \geq 1) \in \ell^2_\downarrow$,

$$\operatorname{ord}(\zeta(\varepsilon^n_k), k \geq 1) \xrightarrow{d} (\zeta(\widetilde{\varepsilon}_i), i \geq 1)$$

as $n \to \infty$ *in* $\ell^2_\downarrow$, *and*

$$\operatorname{ord}(\zeta(\varepsilon^n_k), g^n_k, k \geq 1) \xrightarrow{d} (\zeta(\widetilde{\varepsilon}_i), g_i, i \geq 1),$$

where the convergence is in $\ell^2_\downarrow$ *for the first coordinate and in the product topology for the second.*

We apply a method outlined in Proposition 15 of Aldous [13], which is most conveniently recounted in Section 2.6 of Aldous and Limic [15]. This is very similar to Theorem 8.3 of Joseph [91], who omits many of the details of the proof. We feel that the argument is sufficiently subtle to merit a full account, which we now give.

Essentially, there are two steps to proving the desired convergence. First, we need to show that the longest excursions of R^n and R occurring before some finite time match up for large enough n. Then we need to show that long excursions of R^n cannot "wander off to time ∞". Proposition 2.5.7 below is designed to deal with these issues.

Following Aldous, we introduce the concept of a size-biased point process. Suppose we have random variables $\mathbf{Y} = (Y_\gamma : \gamma \in \Gamma)$ in $\ell^2_+(\Gamma)$. Given $\mathbf{Y}$, let $E_\gamma \sim \operatorname{Exp}(Y_\gamma)$ independently for different $\gamma \in \Gamma$. Set

$$\Sigma(a) = \sum_{\gamma \in \Gamma} Y_\gamma \mathbb{1}_{\{E_\gamma < a\}} \tag{2.19}$$

and note that $\Sigma(a) < \infty$ a.s. Let $\Sigma_\gamma = \Sigma(E_\gamma)$. Then $\Xi = \{(\Sigma_\gamma, Y_\gamma) : \gamma \in \Gamma\}$ is the *size-biased point process* (SBPP) associated with $\mathbf{Y}$. Write π for the projection onto the second co-ordinate, so that $\pi(\{(s_\gamma, y_\gamma)\}) = \{y_\gamma\}$.

Proposition 2.5.7 (Proposition 15 of [13] and Proposition 17 of [15])**.** *Let* $\mathbf{Y}^n \in \ell^2_+(\Gamma^n)$ *for each* $n > 1$, *let* Σ^n *be the analogue of (2.19) and let* Ξ^n *be the associated SBPP. Suppose that*

$$\Xi^n \xrightarrow{d} \Xi^\infty,$$

as $n \to \infty$, *for the topology of vague convergence of counting measures on* $[0, \infty) \times (0, \infty)$, *where* Ξ^∞ *is some point process satisfying*

1. $\sup\{s : (s, y) \in \Xi^\infty$ *for some* $y\} = \infty$ *a.s.*
2. *if* $(s, y) \in \Xi^\infty$ *then* $\sum_{(s', y') \in \Xi^\infty : s' < s} y' = s$ *a.s.*
3. $\max\{y : (s, y) \in \Xi^\infty$ *for some* $s > t\} \xrightarrow{p} 0$ *as* $t \to \infty$.

Then $\mathbf{Y}^\infty := \operatorname{ord}(\pi(\Xi^\infty)) \in \ell^2_\downarrow$ *and* $\operatorname{ord}(\mathbf{Y}^n) \xrightarrow{d}$ *ord*$(\mathbf{Y}^\infty)$ *in* $\ell^2_\downarrow$. *In addition,*

$$\operatorname{ord}(Y^n_\gamma, \Sigma^n_\gamma, \gamma \in \Gamma^n) \xrightarrow{d} \operatorname{ord}(Y_\gamma, \Sigma_\gamma, \gamma \in \Gamma)$$

as $n \to \infty$, *where the convergence is in* $\ell^2_\downarrow$ *for the first coordinate and in the product topology for the second.*

The original statement does not mention the last convergence. It is, in fact, implicitly contained in the proof of Proposition 15 given in [13], more precisely in the assertion that the tightness of the sequence $(\Sigma^n(a))_{n\geq 1}$ for arbitrary $a > 0$ and the convergence $\Xi^n \xrightarrow{d} \Xi^\infty$ together imply the convergence $\text{ord}(\mathbf{Y}^n) \xrightarrow{d} \text{ord}(\mathbf{Y}^\infty)$ in ℓ^2.

For $k \geq 1$, we let

$$Y_k^n = n^{-\alpha/(\alpha+1)} \left[\zeta^n(k) - (N^n(\sigma^n(k)) - N^n(\sigma^n(k-1)) + 1)\right].$$

Recall that at the end of the exploration of the $(k-1)$th component of $\mathbf{M}_n(\nu)$ in depth-first order, we choose a new vertex from the unexplored parts of the graph with probability proportional to its degree. So we pick a *component* with probability proportional to the sum of its degrees, which is twice the number of its edges. Since the number of steps it takes to explore a component of $\mathbf{M}_n(\nu)$ is the number of its vertices (which is the number of its non-back-edges plus one) plus twice the number of back-edges (unlike the non-back-edges, a back-edge takes one step to create and another step to close), it follows that Y_k^n is the number of edges of the kth component of $\mathbf{M}_n(\nu)$ times $n^{-\alpha/(\alpha+1)}$.

For $k \geq 1$, let

$$\Sigma_k^n = \sum_{i=1}^{k-1} Y_i^n = n^{-\alpha/(\alpha+1)} \left[\sigma^n(k-1) - N^n(\sigma^n(k-1)) - k + 1\right],$$

and put

$$\Xi^n = \{(\Sigma_k^n, Y_k^n) : k \geq 1\}.$$

It is easy to see that Ξ^n then has the same distribution as the SBPP associated with $n^{-\alpha/(\alpha+1)}(\zeta^n(k) - N^n(\sigma^n(k)) + N^n(\sigma^n(k-1)) - 1, k \geq 1)$.

Recall from Section 2.3 the notation $(\widetilde{\varepsilon}^{(\ell)}, \ell \geq 0)$ for the excursions of the reflected process $(R_t, t \geq 0)$ indexed by local time ℓ, and $(\widetilde{\sigma}_\ell, \ell \geq 0)$ for the inverse local time process. Let Ξ^∞ be the point process given by

$$\Xi^\infty = \{(\widetilde{\sigma}_{\ell-}, \zeta(\widetilde{\varepsilon}^{(\ell)})) : \ell \geq 0, \widetilde{\sigma}_\ell - \widetilde{\sigma}_{\ell-} > 0\}.$$

By Proposition 2.3.1 and Lemma 2.3.5, properties (1), (2) and (3) from Proposition 2.5.7 above hold for Ξ^∞. In order to apply Proposition 2.5.7, it thus remains to establish the convergence of Ξ^n to Ξ^∞. We do this by first proving a deterministic result for a suitable class of functions, extending Lemma 7 of Aldous [13] from the setting of continuous functions to the setting of càdlàg functions satisfying certain conditions.

For a càdlàg function $f : [0, \infty) \to \mathbb{R}$ with only positive jumps, let $E(f)$ be the set of non-empty intervals $e = (l, r)$ such that $f(l) = \inf_{s \leq l} f(s) = f(r)$ and $f(s) > f(l)$ for all $s \in (l, r)$. We say that such intervals are excursions of f. Let $\mathcal{S}$ denote the set of functions $f : [0, \infty) \to \mathbb{R}$ satisfying the following conditions:

1. f is càdlàg and has only non-negative jumps.

2. $f(x) \to -\infty$ as $x \to \infty$.

3. If $0 \leq a < b$ and $f(b-) = \inf_{a \leq s \leq b} f(s)$ then $f(b) = f(b-)$.

4. For each $\epsilon > 0$, $E(f)$ contains only finitely many excursions of length greater than or equal to ϵ.

5. The complement of $\cup_{(l,r) \in E(f)} (l, r)$ has Lebesgue measure 0.

6. If $(l_1, r_1), (l_2, r_2) \in E(f)$ and $l_1 < l_2$, then $f(l_1) > f(l_2)$.

Lemma 2.5.8. *Let $f \in \mathcal{S}$ and let $(f_n)_{n \geq 1}$ be a sequence of càdlàg functions such that $f_n \to f$ as $n \to \infty$ in the Skorokhod sense. For each $n \in \mathbb{N}$, let $(t_{n,i})_{i \geq 1}$ be a strictly increasing sequence such that*

(i) $t_{n,1} = 0$ *and* $\lim_{i \to \infty} t_{n,i} = \infty$,

(ii) $f_n(t_{n,i}) = \inf_{s \leq t_{n,i}} f_n(s)$,

(iii) *for each* $s < \infty$, $\lim_{n \to \infty} \max_{i: t_{n,i} \leq s} (f_n(t_{n,i}) - f_n(t_{n,i+1})) = 0$.

Write $\Xi = \{(l, r - l) : (l, r) \in E(f)\}$ *and* $\Xi_n = \{(t_{n,i}, t_{n,i+1} - t_{n,i}) : i \geq 1\}$ *for* $n \geq 1$. *Then*

$$\Xi_n \to \Xi$$

as $n \to \infty$, where the convergence holds in the topology of vague convergence of counting measures on $[0, \infty) \times (0, \infty)$.

Proof. We adapt the proof of Lemma 4.8 of Martin & Ráth [108]. Suppose that (l, r) is an excursion of f. Fix $\epsilon \in (0, r - l)$. Since $f \in \mathcal{S}$, there exists $\delta > 0$ such that

$$f(x) \geq f(l) + \delta \text{ for all } x \in [0, l - \epsilon/2]$$

$$f(x) \geq f(l) + \delta \text{ for all } x \in [l + \epsilon/2, r - \epsilon/2]$$

$$f(x) \leq f(l) - \delta \text{ for some } x \in (r, r + \epsilon/2].$$

The first line is a consequence of conditions (1) and (6) in the definition of the set $\mathcal{S}$: if for every $n > 0$, there exists $x_n \in [0, l - \epsilon/2]$ such that $f(x_n) < f(l) + 1/n$, then by condition (1) there would be an accumulation point $x_\infty \in [0, l - \epsilon/2]$ of the sequence $(x_n)_{n \geq 1}$ such that $f(x_\infty) \leq f(l)$, and hence there would exist an interval (l', r') with $r' < r$ such that $f(r') \leq f(x_\infty) \leq f(r)$. But this contradicts (6).

The second line follows from a similar argument (there cannot be an accumulation point $x_\infty \in [l + \epsilon/2, r - \epsilon/2]$ such that $f(x_\infty) \leq f(l)$ since (l, r) is an excursion interval). The third line is again a consequence of condition (6) in the definition of the set $\mathcal{S}$.

Since we have $f_n \to f$ in the Skorokhod sense, there exist n_0 and a sequence of continuous strictly increasing functions $\lambda_n : [0,\infty) \to [0,\infty)$ such that $\lambda_n(0) = 0$, $\lim_{t\to\infty} \lambda_n(t) = \infty$ and for all $n \geq n_0$,

$$|f_n(\lambda_n(x)) - f(x)| < \delta/2 \quad \text{for all } x \in [0, r + \epsilon/2]$$

and

$$|\lambda_n(x) - x| < \epsilon/2 \quad \text{for all } x \in [0, r + \epsilon/2].$$

Then for $n \geq n_0$,

$f_n(\lambda_n(x)) \geq f(l) + \delta/2$ for all $x \in [0, l - \epsilon/2]$

$f_n(\lambda_n(x)) > f(l) - \delta/2$ for all $x \in [l - \epsilon/2, l + \epsilon/2]$

$f_n(\lambda_n(l)) < f(l) + \delta/2$

$f_n(\lambda_n(x)) \geq f(l) + \delta/2$ for all $x \in [l + \epsilon/2, r - \epsilon/2]$

$f_n(\lambda_n(x)) \leq f(l) - \delta/2$ for some $x \in [r, r + \epsilon/2]$.

For the second point, note that $f_n(\lambda_n(x)) > f(x) - \delta/2 \geq f(r) - \delta/2 = f(l) - \delta/2$ for every $x \leq l + \epsilon/2 < r$. Therefore,

$f_n(x) \geq f(l) + \delta/2$ for all $x \in [0, l - \epsilon]$

$f_n(x) > f(l) - \delta/2$ for all $x \in [l - \epsilon, l + \epsilon]$

$f_n(x) < f(l) + \delta/2$ for some $x \in [l - \epsilon/2, l + \epsilon/2]$.

$f_n(x) \geq f(l) + \delta/2$ for all $x \in [l + \epsilon, r - \epsilon]$

$f_n(x) \leq f(l) - \delta/2$ for some $x \in [r - \epsilon/2, r + \epsilon]$.

From the first and third inequalities, the set $I := \{x \in [l - \epsilon, r + \epsilon],\ f_n(x) \leq \inf_{y \leq l-\epsilon} f_n(y)\}$ is not empty. Let $l^{(n)} := \inf I$ and let $(x_k)_{k\geq 1}$ be a decreasing sequence in I converging to $l^{(n)}$. Since f_n is right-continuous, $f_n(l^{(n)}) = \lim_{k\to\infty} f_n(x_k)$, and $f_n(l^{(n)}) \leq \inf_{x \leq l-\epsilon} f_n(x)$, so that $l^{(n)} = \min I$. Clearly,

$$f_n(l^{(n)}) = \inf_{x \leq l^{(n)}} f_n(x)$$

Similarly, from the first, second, fourth and fifth inequalities, we obtain the existence of $r^{(n)} \in [r - \epsilon, r + \epsilon]$ such that

$$f_n(r^{(n)}) = \inf_{x \leq r^{(n)}} f_n(x).$$

Now fix $\eta > 0$. We can find $t > 0$ such that there are no excursions of length exceeding η in $E(f)$ which intersect $[t, \infty)$. Then there exists a finite collection $\{(l_i, r_i) : 1 \leq i \leq m\}$ of excursions in $E(f)$ with $l_i \leq t + \eta$ for $1 \leq i \leq m$ and such that $\cup_{1\leq i\leq m}(l_i, r_i)$ covers all of

$[0, t+\eta]$ except for a set of Lebesgue measure at most $\eta/2$. Set $\epsilon = \eta/4m$ and apply the above argument for each excursion, to see that for n sufficiently large, there exist disjoint intervals $(l_1^{(n)}, r_1^{(n)}), \ldots, (l_m^{(n)}, r_m^{(n)})$ such that

$$|l_i^{(n)} - l_i| < \eta/4m \quad \text{and} \quad |r_i^{(n)} - r_i| < \eta/4m \quad \text{for } 1 \le i \le m.$$

But then the remaining length in $[0, t+\eta]$ is at most η and so, in particular, we must have captured all possible intervals $(t_{n,i}, t_{n,i+1})$ with $t_{n,i} \le t+\eta$ and $t_{n,i+1} - t_{n,i} \ge \eta$, up to an error of at most $\eta/4m$ at each end-point. The required vague convergence follows straightforwardly. □

Lemma 2.5.9. *We have $\widetilde{L} \in \mathcal{S}$ almost surely. Moreover,*

$$\Xi^n \xrightarrow{d} \Xi^\infty$$

as $n \to \infty$, where the convergence holds in the topology of vague convergence of counting measures on $[0, \infty) \times (0, \infty)$.

Proof. $\widetilde{L}$ is clearly càdlàg with only non-negative jumps almost surely. The other conditions required for a function to lie in $\mathcal{S}$ follow from Proposition 2.3.1 and Lemma 2.3.5.

Now let $f = \widetilde{L}$ and $f_n = n^{-1/(\alpha+1)} X^n(\lfloor n^{\alpha/(\alpha+1)} \cdot \rfloor)$. It is clear that Ξ^∞ is Ξ of the previous lemma for this f. For $n \ge 1$, $i \ge 0$, let $t_{n,i} = n^{-\alpha/(\alpha+1)} \sigma^n(i)$. Then $t_{n,0} = 0$ and $\lim_{i\to\infty} t_{n,i} = \infty$. By construction, the $t_{n,i}$ are times at which new infima of f_n are reached. Moreover,

$$f_n(t_{n,i}) - f_n(t_{n,i+1}) = n^{-1/(\alpha+1)}(X^n(\sigma^n(i)) - X^n(\sigma^n(i+1))) = n^{-1/(\alpha+1)}.$$

Hence, the $(t_{n,i})_{i\ge1}$ satisfy the conditions in Lemma 2.5.8. It follows that

$$\Xi_n = \{(t_{n,i}, t_{n,i+1} - t_{n,i}), i \ge 1\} \xrightarrow{d} \Xi$$

as $n \to \infty$. Now we have

$$\Sigma_i^n = t_{n,i} - n^{-\alpha/(\alpha+1)}[N^n(\sigma^n(i)) + i]$$

and

$$Y_i^n = t_{n,i} - t_{n,i-1} - n^{-\alpha/(\alpha+1)}[N^n(\sigma^n(i)) - N^n(\sigma^n(i-1)) + 1].$$

Since $n^{-\alpha/(\alpha+1)} N^n(\sigma^n(i)) \to 0$ for each $i \ge 0$, it is straightforward to see that Ξ^n and $\{(t_{n,i}, t_{n,i+1} - t_{n,i}), i \ge 1\}$ can be made arbitrarily close in the vague topology by taking n large. Hence, $\Xi^n \xrightarrow{d} \Xi^\infty$ as desired. □

Proposition 2.5.7 tells us that we may now extract the ordered excursion lengths, and that we can add the convergence of the starting points of the excursions. This completes the proof of Proposition 2.5.6. As an aside, we observe that Proposition 2.5.6 gives us an analogue of Corollary 16 of [13], as follows.

Corollary 2.5.10. *The point process $\Xi^\infty = \{(\widetilde{\sigma}^{(\ell)}, \zeta(\widetilde{\varepsilon}^{(\ell)})) : \ell \ge 0, \widetilde{\sigma}_\ell - \widetilde{\sigma}_{\ell-} > 0\}$ consisting of the left endpoints and the lengths of the excursions of R is distributed as the SBPP associated with the set of excursion lengths of R.*

2.5.5 Marked excursions

We now strengthen the convergence in Proposition 2.5.6 to a convergence of ordered marked excursions. Let us first prove a deterministic analytic result, similar in spirit to Lemma 2.5.8, which we shall use to handle the positions of the back-edges (recall that when a back-edge is discovered, its other endpoint is explored at the first time when the corresponding mark in the stack reaches the top of the stack).

Lemma 2.5.11. *Let $f \in \mathcal{S}$ and let $(f_n)_{n\geq 1}$ be a sequence of càdlàg functions such that $f_n \to f$ as $n \to \infty$ in the Skorokhod sense. For each $n \in \mathbb{N}$, let (s_n, y_n) be a pair of points such that $s_n \geq 0$ and $y_n \leq f_n(s_n)$. Let (s, y) be such that $s > 0$, $f(s-) = f(s)$ and $0 < y < f(s)$. Let $t_n = \inf\{u \geq s_n : f_n(u) \leq y_n\}$ and $t = \inf\{u \geq s : f(u) \leq y\}$. Suppose that $(s_n, y_n) \to (s, y)$, that t is not a local minimum of f and that $f(t-) = f(t)$. Then $t_n \to t$ as $n \to \infty$.*

Proof. Fix $0 < \epsilon < t - s$. Since $f \in \mathcal{S}$, $y < f(s)$, $f(s-) = f(s)$, $f(t-) = f(t)$ and t is not a local minimum of f, there exists $\delta > 0$ such that

$$
\begin{aligned}
f(x) &\geq y + \delta \quad \text{for all } x \in [s - \epsilon/2, t - \epsilon/2] \\
f(x) &\leq y - \delta \quad \text{for some } x \in (t, t + \epsilon/2].
\end{aligned}
$$

As $f_n \to f$, there exist n_0 and a sequence of continuous strictly increasing functions $\lambda_n : [0, \infty) \to [0, \infty)$ such that $\lambda_n(0) = 0$, $\lim_{x\to\infty} \lambda_n(x) = \infty$ and, for all $n \geq n_0$,

$$
|f_n(\lambda_n(x)) - f(x)| < \delta/2 \quad \text{for all } x \in [0, t + \epsilon/2]
$$

and

$$
|\lambda_n(x) - x| < \epsilon/4 \quad \text{for all } x \in [0, t + \epsilon].
$$

Then

$$
\begin{aligned}
f_n(\lambda_n(x)) &\geq y + \delta/2 \quad \text{for all } x \in [s - \epsilon/2, t - \epsilon/2] \\
f_n(\lambda_n(x)) &\leq y - \delta/2 \quad \text{for some } x \in (t, t + \epsilon/2].
\end{aligned}
$$

By taking n_0 larger if necessary, we also have

$$
|s - s_n| < \epsilon/4 \quad \text{and} \quad |y - y_n| < \delta/2
$$

for all $n \geq n_0$. Then for all $n \geq n_0$, we have $|\lambda_n^{-1}(s_n) - s| < \epsilon/2$ and so

$$
f_n(x) \geq y + \delta/2 \quad \text{for all } x \in [s_n, t - \epsilon].
$$

It follows that

$$
f_n(x) > y_n \quad \text{for all } x \in [s_n, t - \epsilon].
$$

Moreover, it must be the case that f_n goes below y_n in the time-interval $[\lambda_n(t-\epsilon/2), \lambda_n(t+\epsilon/2)]$, i.e. we must have $t_n \in [\lambda_n(t - \epsilon/2), \lambda_n(t + \epsilon/2)]$. But then $t_n \in [t - \epsilon, t + \epsilon]$ for all $n \geq n_0$. As ϵ was arbitrary, the result follows. □

Let

$$M^n(k) = N^n(\sigma^n(k)) - N^n(\sigma^n(k-1)),$$

be the number of back-edges falling in the excursion ε_k^n. Suppose that $M^n(k) \geq 1$. Then for $1 \leq r \leq M^n(k)$, let $g_k^n + s_{k,r}^n$ be the rescaled time at which the rth back-edge is discovered in the kth component and let $n^{1/(\alpha+1)} x_{k,r}^n \geq 0$ be its position on the stack. Let $g_k^n + t_{k,r}^n$ be the rescaled time at which the corresponding marked leaf in the stack is killed, thus closing the rth back-edge. It is the first time that the stack size goes below the height of that leaf after its discovery (the stack being a LIFO queue), so that we have

$$t_{k,r}^n = \inf\{t \geq s_{k,r}^n : \varepsilon_k^n(t) \leq x_{k,r}^n - n^{-1/(\alpha+1)}\}.$$

Finally, if $M^n(k) \geq 1$, let $\mathcal{P}_k^n = \sum_{r=1}^{M^n(k)} \delta_{(s_{k,r}^n, t_{k,r}^n)}$ define a point measure on $[0, \zeta(\varepsilon_k^n)]^2$. If $M^n(k) = 0$, let $\mathcal{P}_k^n$ be the null measure. Let

$$\mathcal{Q}^n = \sum_{k \geq 1} \sum_{r=1}^{M^n(k)} \delta_{(g_k^n + s_{k,r}^n, g_k^n + t_{k,r}^n)},$$

the point measure encompassing all of the pairs of rescaled times at which a back-edge is opened and closed.

Turning now to the limiting process, recall that $(\widetilde{\varepsilon}_i, i \geq 1)$ are the excursions of R listed in decreasing order of length, and that the sequence $(\zeta(\widetilde{\varepsilon}_i), i \geq 1)$ lies in $\ell^2_\downarrow$. Let M_i be the number of marks falling in the excursion $\widetilde{\varepsilon}_i$, and if $M_i \geq 1$, write $s_{i,1}, \ldots, s_{i,M_i}$ for the times and $x_{i,1}, \ldots, x_{i,M_i}$ for the positions of the marks, respectively. For $1 \leq r \leq M_i$, let

$$t_{i,r} = \inf\{t \geq s_{i,r} : \widetilde{\varepsilon}_i(t) \leq x_{i,r}\}.$$

If $M_i \geq 1$, write $\mathcal{P}_i = \sum_{r=1}^{M_i} \delta_{(s_{i,r}, t_{i,r})}$, and if $M_i = 0$, let $\mathcal{P}_i$ be the null measure. Finally, let

$$\mathcal{Q} = \sum_{i \geq 1} \sum_{r=1}^{M_i} \delta_{(g_i + s_{i,r}, g_i + t_{i,r})}.$$

Recall that

$$\widetilde{\sigma}^n(i) = \min\{k : \widetilde{S}^n(k) \leq -i\}$$

and let $\widetilde{\zeta}^n(i) = \widetilde{\sigma}^n(i) - \widetilde{\sigma}^n(i-1)$ be the length of the ith excursion of $\widetilde{S}^n$ above its running minimum. Since the components of $\widetilde{\mathbf{F}}_n(\nu)$ again appear in size-biased order, an argument completely analogous to that above gives that $n^{-\alpha/(\alpha+1)} \mathrm{ord}(\widetilde{\zeta}^n(k), k \geq 1)$ converges in distribution to $(\zeta(\widetilde{\varepsilon}_i), i \geq 1)$ in $\ell^2_\downarrow$. In particular,

$$n^{-\alpha/(\alpha+1)} \max_{i \geq 1} \widetilde{\zeta}^n(i) \xrightarrow{d} \zeta(\widetilde{\varepsilon}_1),$$

where we recall that $\zeta(\widetilde{\varepsilon}_1)$ is the length of the longest excursion of R above 0; in particular, by Proposition 2.3.1 and Lemma 2.3.5 we have $\zeta(\widetilde{\varepsilon}_1) < \infty$ a.s.

For $i \geq 1$, let $\widetilde{M}^n(i)$ be the number of back-edges falling among the vertices corresponding to the $(2i-1)$th and $2i$th components of $\widetilde{\mathbf{F}}_n(\nu)$, and let $\widetilde{N}^n_{\max} = \max_{k\geq 1} \widetilde{M}^n(k)$. On the pair formed of the $(2i-1)$th and $2i$th components of $\widetilde{\mathbf{F}}_n(\nu)$, at corresponding vertices, we have that the size of the stack in $\mathbf{F}_n(\nu)$ is bounded above by the size of the stack in $\widetilde{\mathbf{F}}_n(\nu)$ plus 1.

Proposition 2.5.12. *We have*

$$\operatorname{ord}\left(\zeta(\varepsilon^n_k), \varepsilon^n_k, \mathcal{P}^n_k, k \geq 1\right) \xrightarrow{d} \left(\zeta(\widetilde{\varepsilon}_i), \widetilde{\varepsilon}_i, \mathcal{P}_i, i \geq 1\right)$$

as $n \to \infty$. Here, for each $k \geq 1$, the convergence in the second co-ordinate is for the Skorokhod topology and in the third for the Hausdorff distance on $\mathbb{R}^2_+$; then we take the product topology over the different indices.

Proof. Because the points $\{(s_{i,r}, x_{i,r}) : 1 \leq r \leq M_i\}$ are picked uniformly from the Lebesgue measure under the excursion $\widetilde{\varepsilon}_i$ (recall that these points are the marks of a uniform Cox process under R, Theorem 2.5.5), for each $i \geq 1$, and such an excursion has only countably many discontinuities (all of which are up-jumps), the conditions of Lemma 2.5.11 are fulfilled almost surely. Combining this with Theorem 2.5.5, we may deduce the joint convergence in distribution as $n \to \infty$ of

$$(n^{-1/(\alpha+1)} R^n(\lfloor t n^{\alpha/(\alpha+1)} \rfloor), t \geq 0) \to (R_t, t \geq 0)$$

in the Skorokhod topology and $\mathcal{Q}^n \to \mathcal{Q}$ in the topology of vague convergence on $\mathbb{R}^2_+$.

By Skorokhod's representation theorem, we may work on a probability space such that this joint convergence holds almost surely. Fix $K \in \mathbb{N}$. We have already shown that, given $\delta > 0$, there exist $M > 0$ and n_0 sufficiently large such that, with probability at least $1 - \delta$, the K longest excursions of both $n^{-1/(\alpha+1)} R^n(\lfloor \cdot\, n^{\alpha/(\alpha+1)} \rfloor)$ and R occur in the time-interval $[0, M]$ for all $n \geq n_0$. Since $\mathcal{Q}^n \to \mathcal{Q}$ and $\mathcal{Q}$ has only finitely many points in $[0, M]^2$, we may deduce that the K smaller point processes obtained by restricting $\mathcal{Q}^n$ to each of the K longest excursion-intervals converge in the sense of the Hausdorff distance to $\mathcal{P}_1, \ldots, \mathcal{P}_K$ respectively. The result follows. □

We conclude this section with some technical bounds on the number of back-edges in a given component, of which we will make use later.

Lemma 2.5.13. *Almost surely, we have $\#\{i \geq 1 : M_i \geq 2\} < \infty$ and $N_{\max} := \sup_{i \geq 1} M_i < \infty$.*

Proof. We will bound

$$\mathbb{E}\left[\#\{i \geq 1 : M_i \geq 2\} \middle| (\zeta(\widetilde{\varepsilon}_i))_{i \geq 1}\right] = \sum_{i \geq 1} \mathbb{P}\left(M_i \geq 2 \middle| \zeta(\widetilde{\varepsilon}_i)\right),$$

where we have used the independence of the excursions given their lengths. We use the crude bound $\mathbb{P}(\text{Po}(\lambda) \geq 2) \leq \lambda^2$. For any $i \geq 1$, we have

$$\mathbb{P}\left(M_i \geq 2 \middle| \zeta(\widetilde{\varepsilon}_i) = x\right) = \mathbb{P}\left(\text{Po}\left(\frac{1}{\mu}\int_0^x \widetilde{\varepsilon}_i(u)du\right) \geq 2 \;\middle|\; \zeta(\widetilde{\varepsilon}_i) = x\right)$$
$$\leq \frac{\mathbb{E}_{\mathbb{N}^{(x)}}\left[\left(\frac{1}{\mu}\int_0^x \mathbb{e}(u)du\right)^2 \exp\left(\frac{1}{\mu}\int_0^x \mathbb{e}(u)du\right)\right]}{\mathbb{E}_{\mathbb{N}^{(x)}}\left[\exp\left(\frac{1}{\mu}\int_0^x \mathbb{e}(u)du\right)\right]}$$
$$\leq \mathbb{E}_{\mathbb{N}^{(x)}}\left[\left(\frac{1}{\mu}\int_0^x \mathbb{e}(u)du\right)^4\right]^{1/2} \mathbb{E}_{\mathbb{N}^{(x)}}\left[\exp\left(\frac{2}{\mu}\int_0^x \mathbb{e}(u)du\right)\right]^{1/2},$$

by the Cauchy–Schwarz inequality and the fact that the denominator is bounded below by 1.

By the scaling property of the excursion $\mathbb{e}$,

$$\mathbb{E}_{\mathbb{N}^{(x)}}\left[\left(\frac{1}{\mu}\int_0^x \mathbb{e}(u)du\right)^4\right]^{1/2} \leq Cx^{2(1+1/\alpha)}$$

for some constant $C > 0$. Define

$$f(x) := \mathbb{E}_{\mathbb{N}^{(x)}}\left[\exp\left(\frac{2}{\mu}\int_0^x \mathbb{e}(u)du\right)\right]^{1/2} = \mathbb{E}_{\mathbb{N}^{(1)}}\left[\exp\left(\frac{2x^{1+1/\alpha}}{\mu}\int_0^1 \mathbb{e}(u)du\right)\right]^{1/2}.$$

This is clearly an increasing function of x, so that for $x \leq \zeta(\widetilde{\varepsilon}_1)$ we have $f(x) \leq f(\zeta(\widetilde{\varepsilon}_1))$, which is almost surely finite by Lemma 2.3.6, since $\zeta(\widetilde{\varepsilon_1}) < \infty$ a.s. By Proposition 2.5.6, we have $\sum_{i\geq 1} \zeta(\widetilde{\varepsilon}_i)^2 < \infty$ a.s. and so

$$\mathbb{E}\left[\#\{i \geq 1 : M_i \geq 2\} \middle| (\zeta(\widetilde{\varepsilon}_i))_{i\geq 1}\right] \leq f(\zeta(\widetilde{\varepsilon}_1)) \sum_{i\geq 1} \zeta(\widetilde{\varepsilon}_i)^{2(1+1/\alpha)} < \infty \quad \text{a.s.}$$

It follows that

$$\#\{i \geq 1 : M_i \geq 2\} < \infty \quad \text{a.s.}$$

Since the area of any individual excursion $\widetilde{\varepsilon}_i$ is finite, it contains an almost surely finite number of points and this, together with the fact that only finitely many excursions contain more than 2 points, gives that $N_{\max} < \infty$ a.s. □

2.5.6 Height process

In order to deal with the metric structure, we also need to know that the height process associated with X^n converges. Let

$$H^n(k) = \#\left\{j \in \{0, 1, \ldots, k-1\} : X^n(j) = \inf_{j\leq \ell\leq k} X^n(\ell)\right\}$$

so that $H^n(k)$ is the distance from the root of the component being explored to the current vertex at step k. See Figure 2.2 for an illustration. (Recall that $\mathbf{M}_n(\nu)$ is obtained from $\mathbf{F}_n(\nu)$ by replacing pairs of marked leaves by back-edges.)

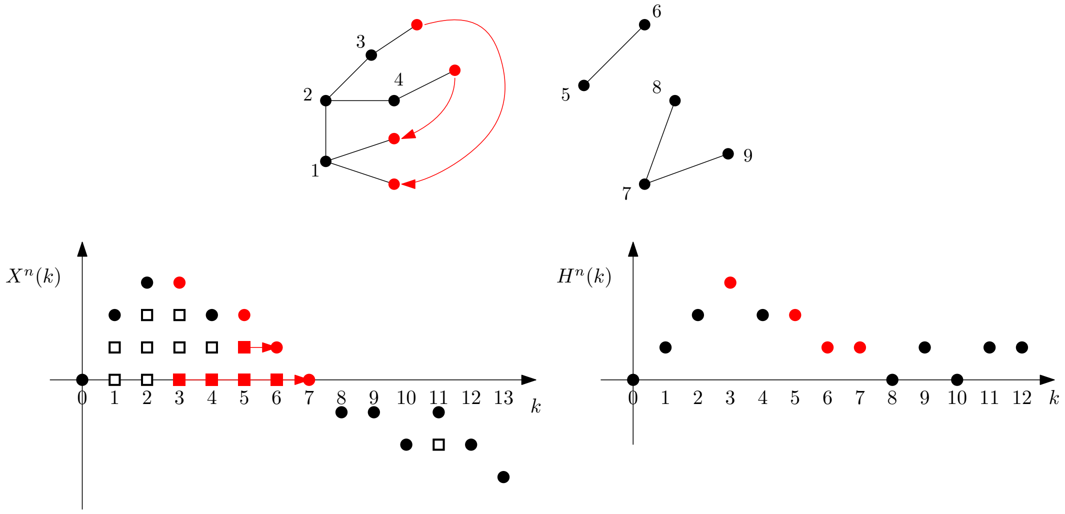

Figure 2.2: Top: three components of $\mathbf{F}_n(\nu)$. Left: $X^n(k)$ drawn with the vertices on the stack indicated. Empty squares represent half-edges which are available to be connected to as back-edges. Filled squares are marked vertices. Right: the corresponding height process (with the extra vertices in red).

Our aim in this section is to prove that H^n has $\widetilde{H}$ as its scaling limit, and that we can extract its marked excursions as for the exploration process. Recall that

$$\widetilde{G}^n(k) = \#\left\{ j \in \{0, 1, \dots, k-1\} : \widetilde{S}^n(j) = \inf_{j \le \ell \le k} \widetilde{S}^n(\ell) \right\},$$

which is the height process corresponding to the forest $\widetilde{\mathbf{F}}_n(\nu)$. We will compare H^n and $\widetilde{G}^n$. It will be sufficient to do this for pairs of components of $\widetilde{\mathbf{F}}_n(\nu)$. To this end, suppose that for $a \ge 0$ and $m \ge 1$, vertices $v_{a+1}, v_{a+2}, \dots, v_{a+m}$ form a pair of components of $\widetilde{\mathbf{F}}_n(\nu)$, and that there are b back-edges on these vertices in $\mathbf{M}_n(\nu)$. Then the corresponding collection of components in $\mathbf{F}_n(\nu)$ together have $m+2b$ vertices, which are visited at times $c+a, c+a+1, \dots, c+a+m+2b-1$ in the depth-first exploration, where $c \ge 0$ is such that $a = \tau_n(c+a)$. We compare H^n with $\widetilde{G}^n$ at the times $\tau_n(c+a), \tau_n(c+a+1), \dots, \tau_n(c+a+m+2b-1)$.

Lemma 2.5.14. *We have*

$$\max_{0 \le i \le m+2b-1} |H^n(a+c+i) - \widetilde{G}^n(\tau_n(a+c+i))| \le 1 + b + 2b \max_{1 \le i \le m} |\widetilde{G}^n(a+i) - \widetilde{G}^n(a+i-1)|$$

Proof. Suppose first that $b = 0$. Until we come to the end of the first component of $\widetilde{\mathbf{F}}_n(\nu)$, we have

$$H^n(a+i) = \widetilde{G}^n(\tau_n(a+i)).$$

Thereafter, we have $H^n(a+i) = 1 + \widetilde{G}^n(\tau_n(a+i))$, since the second component of $\widetilde{\mathbf{F}}_n(\nu)$ becomes a subtree attached to the root of the first component of $\mathbf{F}_n(\nu)$.

For $b > 0$, we encourage the reader to refer back to Figure 2.1.

Write $\Delta = \max_{1\le i\le m} |\widetilde{G}^n(a+i) - \widetilde{G}^n(a+i-1)|$. The occurrence of a head or tail of a back-edge at time k implies $\tau_n(k+1) = \tau_n(k)$. To get from $\widetilde{\mathbf{F}}_n(\nu)$ to $\mathbf{F}_n(\nu)$, we unplug a sequence of subtrees and plug them back in further along in the depth-first order. Within subtrees containing no back-edges, the *increments* of the height process remain the same as they are in the corresponding subtrees of $\widetilde{\mathbf{F}}_n(\nu)$. Finally, every time we start a new subtree there is an extra difference of 1. It follows that the most by which the height process can be altered in going from $\widetilde{\mathbf{F}}_n(\nu)$ to $\mathbf{F}_n(\nu)$ is an additive factor of $1 + b + 2b\Delta$. □

We are now ready to state and prove the main result of this section.

Proposition 2.5.15. *Jointly with the convergence in Theorem 2.5.5, we have that as $n \to \infty$,*

$$\left(n^{-\frac{\alpha-1}{\alpha+1}} H^n(\lfloor tn^{\alpha/(\alpha+1)}\rfloor), t \ge 0\right) \xrightarrow{d} \left(\widetilde{H}_t, t \ge 0\right)$$

in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$.

Proof. By Theorem 2.4.1, we have

$$\left(n^{-\frac{\alpha-1}{\alpha+1}} \widetilde{G}^n(\lfloor n^{\frac{\alpha}{\alpha+1}} u\rfloor), u \ge 0\right) \xrightarrow{d} (\widetilde{H}_u, u \ge 0).$$

By Theorem 1.4.3 of Duquesne and Le Gall [70], $(H_u, u \ge 0)$ is almost surely continuous. By the absolute continuity in Proposition 2.3.2, the same is true of $(\widetilde{H}_u, 0 \le u \le t)$. It follows that for any $t > 0$,

$$n^{-\frac{\alpha-1}{\alpha+1}} \sup_{1\le j\le \lfloor tn^{\alpha/(\alpha+1)}\rfloor} |\widetilde{G}^n(j) - \widetilde{G}^n(j-1)| \xrightarrow{d} 0 \tag{2.20}$$

as $n \to \infty$.

By Lemma 2.5.13, if $\delta > 0$, there exists $K_\delta < \infty$ such that

$$\mathbb{P}\left(\zeta(\widetilde{\varepsilon}_1) > K_\delta\right) < \delta \qquad \text{and} \qquad \mathbb{P}\left(N_{\max} > K_\delta\right) < \delta.$$

In particular, starting from any time k, the number of steps until we next reach the beginning of an odd-numbered component is bounded above by $2K_\delta n^{\alpha/(\alpha+1)}$ with probability at least $1 - \delta + o(1)$.

Now fix $t > 0$ and $\epsilon > 0$. Then by Lemma 2.5.14, we have

$$\begin{aligned}
&\mathbb{P}\left(n^{-\frac{\alpha-1}{\alpha+1}} \sup_{0\le k\le \lfloor tn^{\alpha/(\alpha+1)}\rfloor} |H^n(k) - \widetilde{G}^n(\tau_n(k))| > \epsilon\right) \\
&\quad \le \mathbb{P}\left(n^{-\alpha/(\alpha+1)} \max_{i\ge 1} \widetilde{\zeta}^n(i) > K_\delta\right) + \mathbb{P}\left(N^n_{\max} > K_\delta\right) \\
&\quad + \mathbb{P}\left(1 + K_\delta + 2K_\delta \sup_{1\le j\le \lfloor (t+2K_\delta) n^{\alpha/(\alpha+1)}\rfloor} |\widetilde{G}^n(j) - \widetilde{G}^n(j-1)| > \epsilon n^{\frac{\alpha-1}{\alpha+1}}\right).
\end{aligned}$$

Using (2.20), we obtain that

$$\mathbb{P}\left(1 + K_\delta + 2K_\delta \sup_{1 \le j \le \lfloor (t+2K_\delta) n^{\alpha/(\alpha+1)} \rfloor} |\widetilde{G}^n(j) - \widetilde{G}^n(j-1)| > \epsilon n^{\frac{\alpha-1}{\alpha+1}}\right) \to 0$$

as $n \to \infty$. It follows that

$$\limsup_{n \to \infty} \mathbb{P}\left(n^{-\frac{\alpha-1}{\alpha+1}} \sup_{0 \le k \le \lfloor t n^{\alpha/(\alpha+1)} \rfloor} |H^n(k) - \widetilde{G}^n(\tau_n(k))| > \epsilon\right) < 2\delta.$$

But $\delta > 0$ was arbitrary and so by Lemma 2.5.1 we may deduce that

$$\left(n^{-\frac{\alpha-1}{\alpha+1}} H^n(\lfloor u n^{\alpha/(\alpha+1)} \rfloor), 0 \le u \le t\right) \xrightarrow{d} \left(\widetilde{H}_u, 0 \le u \le t\right). \qquad \square$$

Now let

$$h_k^n(t) = n^{-(\alpha-1)/(\alpha+1)} H^n(\sigma^n(k-1) + \lfloor t n^{\alpha/(\alpha+1)} \rfloor), \quad 0 \le t \le n^{-\alpha/(\alpha+1)} \zeta^n(k).$$

Proposition 2.5.16. *We have*

$$\operatorname{ord}(\varepsilon_k^n, h_k^n, \mathcal{P}_k^n, k \ge 1) \xrightarrow{d} \left(\widetilde{\varepsilon}_i, \widetilde{h}_i, \mathcal{P}_i, i \ge 1\right)$$

as $n \to \infty$. Here, for each $k \ge 1$, the convergence in the first co-ordinate is for the Skorokhod topology and in the second for the topology of vague convergence of counting measures on $[0, \infty)^2$; then we take the product topology over different k.

Proof. We derive this from Proposition 2.5.15 by applying the same reasoning as in the proof of Proposition 2.5.12, using the fact that R^n and H^n have the same excursion-intervals. $\square$

2.5.7 The convergence of the metric structure

We have now assembled all of the ingredients needed in order to prove Theorem 2.1.1. Recall that $M_1^n, M_2^n, \dots$ are the components of the random multigraph $\mathbf{M}_n(\nu)$, listed in decreasing order of size. We will make the distance and measure explicit in each by writing $(M_i^n, d_i^n, \mu_i^n)_{i \ge 1}$. Recall also that $\mathbf{F}_n(\nu)$ is the forest encoded by H^n. In order to recover $\mathbf{M}_n(\nu)$ from $\mathbf{F}_n(\nu)$ we remove pairs of marked leaves and replace them by back-edges as described above.

Proof of Theorem 2.1.1. Write $(h_{(i)}^n, \mathcal{P}_{(i)}^n)$ for the ith element of the sequence $\operatorname{ord}(h_k^n, \mathcal{P}_k^n, k \ge 1)$. Write (T_i^n, d_i^n, μ_i^n) for the tree with rescaled height process $h_{(i)}^n$ and with mass $n^{-\alpha/(\alpha+1)}$ on each vertex. (For $i \ge 1$, these are the trees of the forest $\mathbf{F}_n(\nu)$ listed in decreasing order of mass.) By Skorokhod's representation theorem, we may work on a probability space where the convergence

$$\left(h_{(i)}^n, \mathcal{P}_{(i)}^n, i \ge 1\right) \to (\widetilde{h}_i, \mathcal{P}_i, i \ge 1)$$

occurs almost surely. By (2.8), this entails that

$$\mathrm{d}_{\mathrm{GHP}}((T_i^n, d_i^n, \mu_i^n), (\widetilde{\mathcal{T}}_i, \widetilde{d}_i, \widetilde{\mu}_i)) \to 0 \quad \text{a.s.}$$

as $n \to \infty$. In order to obtain (M_i^n, d_i^n, μ_i^n) from (T_i^n, d_i^n, μ_i^n) if $\mathcal{P}_{(i)}^n = \sum_{r=1}^{m_{(i)}^n} \delta_{s_{(i),r}^n, t_{(i),r}^n}$ we must remove the vertices encoded by $s_{(i),r}^n$ and $t_{(i),r}^n$ and replace them by a single edge between their parents, for each $1 \le r \le m_{(i)}^n$. Since edges have rescaled length $n^{-(\alpha-1)/(\alpha+1)} \to 0$ it is straightforward to see (in the same manner as in the Proof of Theorem 22 in [6]) that we get

$$\mathrm{d_{GHP}}((M_i^n, d_i^n, \mu_i^n), (\mathcal{G}_i, d_i, \mu_i)) \to 0 \quad \text{a.s.} \tag{2.21}$$

as $n \to \infty$. The conclusion follows. □

2.6 Appendix

2.6.1 A change of measure for spectrally positive Lévy processes

Let L be a spectrally positive Lévy process with Lévy measure π satisfying

$$\int_0^\infty (x \wedge x^2)\pi(dx) < \infty. \tag{2.22}$$

Then we may write the Laplace transform of L_t as

$$\mathbb{E}\left[\exp(-\lambda L_t)\right] = \exp(t\Psi(\lambda)),$$

where

$$\Psi(\lambda) = \gamma\lambda + \frac{\delta^2\lambda^2}{2} + \int_0^\infty \pi(dx)(e^{-\lambda x} - 1 + \lambda x).$$

We impose also that

$$\gamma \ge 0, \quad \delta \ge 0 \tag{2.23}$$

and that at least one of the two following conditions holds:

$$\delta > 0 \quad \text{or} \quad \int_0^\infty x\pi(dx) = \infty. \tag{2.24}$$

As observed by Duquesne & Le Gall [70], assumptions (2.22), (2.23) and (2.24) together ensure that L does not drift to $+\infty$ and has paths of infinite variation.

We note that $\int_0^t \Psi(\theta s)ds < \infty$ for all $\theta > 0$ and all $t > 0$.

Lemma 2.6.1. *For any $\theta > 0$, we have*

$$\mathbb{E}\left[\exp\left(-\theta \int_0^t s dL_s\right)\right] = \exp\left(\int_0^t \Psi(\theta s)ds\right) = \mathbb{E}\left[\exp\left(\theta \int_0^t (L_s - L_t)ds\right)\right].$$

In consequence, the process

$$\left(\exp\left(-\theta \int_0^t s dL_s - \int_0^t \Psi(\theta s)ds\right), t \ge 0\right)$$

is a martingale.

Proof. Let $M(ds, dx)$ be a Poisson random measure on $\mathbb{R}_+$ of intensity $ds \otimes \pi(dx)$, and let $\widetilde{M}(ds, dx)$ be its compensated version. Then

$$\mathbb{E}\left[\exp\left(-\theta \int_0^t \int_0^\infty sx\widetilde{M}(ds, dx)\right)\right] = \exp\left(\int_0^t ds \int_0^\infty \pi(dx)(e^{-\theta sx} - 1 + \theta sx)\right).$$

If B is a standard Brownian motion, we obtain

$$\mathbb{E}\left[\exp\left(-\theta \int_0^t sdB_s\right)\right] = \exp\left(\frac{1}{2}\theta^2 \int_0^t s^2 ds\right).$$

Since we may, in general, realise L as

$$L_t = -\gamma t + \delta B_t + \int_0^t \int_0^\infty x\widetilde{M}(ds, dx),$$

where B and $\widetilde{M}$ are independent, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-\theta \int_0^t sdL_s\right)\right] \\
&\quad = \mathbb{E}\left[\exp\left(\theta \int_0^t \gamma sds - \theta \int_0^t \delta sdB_s - \theta \int_0^t \int_0^\infty xs\widetilde{M}(ds, dx)\right)\right] \\
&\quad = \exp\left(\gamma \int_0^t \theta sds + \frac{1}{2}\delta^2 \int_0^t \theta^2 s^2 ds + \int_0^t ds \int_0^\infty \pi(dx)(e^{-\theta sx} - 1 + \theta sx)\right) \\
&\quad = \exp\left(\int_0^t \Psi(\theta s)ds\right).
\end{aligned}$$

The second equality in the statement of the lemma follows on integrating by parts, and the martingale property follows since L has independent increments. □

This martingale plays an important role as a Radon–Nikodym derivative. Fix $\theta > 0$ and consider the process X with independent (but non-stationary) increments and Laplace transform

$$\mathbb{E}\left[\exp(-\lambda X_t)\right] = \exp\left(\int_0^t ds \int_0^\infty \pi(dx)(e^{-\lambda x} - 1 + \lambda x)e^{-\theta xs}\right).$$

(The process X may again be realised as a stochastic integral with respect to a compensated Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$, but this time with intensity $\exp(-xs)ds\pi(dx)$.) Let

$$A_t = -\frac{1}{\theta}\Psi(\theta t) = -\gamma t - \frac{1}{2}\theta\delta^2 t^2 - \frac{1}{\theta}\int_0^\infty \pi(dx)(e^{-\theta tx} - 1 + \theta tx)$$

and $\widetilde{L}_t = \delta B_t + X_t + A_t$. (Note that this is expressed as the Doob–Meyer decomposition of $\widetilde{L}$, with $\delta B_t + X_t$ the martingale part.)

Proposition 2.6.2. *For any $\theta > 0$ and every $t \geq 0$, we have the following absolute continuity relation: for every non-negative integrable functional F,*

$$\mathbb{E}\left[F(\widetilde{L}_s, 0 \leq s \leq t)\right] = \mathbb{E}\left[\exp\left(-\theta \int_0^t sdL_s - \int_0^t \Psi(\theta s)ds\right) F(L_s, 0 \leq s \leq t)\right].$$

Proof. Observe first that Lemma 2.6.1 entails that the change of measure is well-defined for each $t \geq 0$.

Let us first deal with the case where $\gamma = \delta = 0$. We use a decomposition of the Lévy measure similar to that in Bertoin [30] or the proof of Proposition 1 in Miermont [110]:

$$
\begin{aligned}
&\int_0^t ds \int_0^\infty \pi(dx)(e^{-\lambda x} - 1 + \lambda x)e^{-\theta x s} \\
&= \int_0^t ds \int_0^\infty \pi(dx)(e^{-(\lambda+\theta s)x} - 1 + (\lambda + \theta s)x) - \int_0^t ds \int_0^\infty \pi(dx)(e^{-\theta x s} - 1 + \theta x s) \\
&\quad - \int_0^t ds \int_0^\infty \pi(dx)\lambda x(1 - e^{-\theta x s}) \\
&= \int_0^t \Psi(\lambda + \theta s)ds - \int_0^t \Psi(\theta s)ds - \int_0^t ds \int_0^\infty \pi(dx)\lambda x(1 - e^{-\theta x s}).
\end{aligned}
$$

The last integral on the right-hand side makes sense because of the integrability condition (2.22). Indeed, it may be calculated as follows:

$$
\begin{aligned}
\int_0^t ds \int_0^\infty \pi(dx)\lambda x(1 - e^{-\theta x s}) &= \lambda \int_0^\infty x\pi(dx) \int_0^t (1 - e^{-\theta x s})ds \\
&= \frac{\lambda}{\theta} \int_0^\infty \pi(dx)(e^{-\theta t x} - 1 + \theta t x) = \frac{\lambda}{\theta}\Psi(\theta t).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbb{E}\left[\exp(-\lambda X_t)\right] &= \exp\left(\int_0^t ds \int_0^\infty \pi(dx)(e^{-\lambda x} - 1 + \lambda x)e^{-\theta x s}\right) \\
&= \exp\left(-\frac{\lambda}{\theta}\Psi(\theta t) + \int_0^t \Psi(\lambda + \theta s)ds - \int_0^t \Psi(\theta s)ds\right)
\end{aligned}
$$

and we obtain

$$
\mathbb{E}\left[\exp(-\lambda(X_t + A_t))\right] = \exp\left(\int_0^t \Psi(\lambda + \theta s)ds - \int_0^t \Psi(\theta s)ds\right).
$$

Consider the stochastic integral

$$
\int_0^t (\lambda + \theta s)dL_s = \lambda L_t + \theta \int_0^t s dL_s.
$$

We have

$$
\mathbb{E}\left[\exp\left(-\int_0^t (\lambda + \theta s)dL_s\right)\right] = \exp\left(\int_0^t \Psi(\lambda + \theta s))ds\right)
$$

and so

$$
\mathbb{E}\left[\exp(-\lambda(X_t + A_t))\right] = \mathbb{E}\left[\exp\left(-\lambda L_t - \theta \int_0^t s dL_s - \int_0^t \Psi(\theta s)ds\right)\right].
$$

Suppose now that $0 = t_0 < t_1 < \cdots < t_m = t$. Let $\lambda_1, \ldots, \lambda_m \in \mathbb{R}_+$. Then, by the fact that X has independent increments,

$$
\mathbb{E}\left[\exp\left(-\sum_{i=1}^m \lambda_i(\widetilde{L}_{t_i} - \widetilde{L}_{t_{i-1}})\right)\right] = \prod_{i=1}^m \mathbb{E}\left[\exp(-\lambda_i(\widetilde{L}_{t_i} - \widetilde{L}_{t_{i-1}}))\right].
$$

By the same argument as above, we then have

$$\mathbb{E}\left[\exp\left(-\sum_{i=1}^{m}\lambda_i(\widetilde{L}_{t_i}-\widetilde{L}_{t_{i-1}})\right)\right]=\prod_{i=1}^{m}\mathbb{E}\left[\exp\left(-\int_{t_{i-1}}^{t_i}(\lambda_i+\theta s)dL_s-\int_{t_{i-1}}^{t_i}\Psi(\theta s)ds\right)\right]$$
$$=\mathbb{E}\left[\exp\left(-\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(\lambda_i+\theta s)dL_s-\int_0^t\Psi(\theta s)ds\right)\right],$$

since L also has independent increments. Again by integration by parts, we then get that the right-hand side is equal to

$$\mathbb{E}\left[\exp\left(-\sum_{i=1}^{m}\lambda_i(L_{t_i}-L_{t_{i-1}})-\theta\int_0^t sdL_s-\int_0^t\Psi(\theta s)\right)\right].$$

This yields the claimed result for $\gamma=\delta=0$.

Now let us instead suppose that $\gamma\geq 0$, $\delta>0$ and there is no jump component i.e. $L_t=-\gamma t+\delta B_t$ and $\widetilde{L}_t=-\gamma t+\delta B_t-\delta\theta t^2/2$. Then by the Cameron–Martin–Girsanov formula (see, for example, Section 5.6 of Le Gall [96]),

$$\mathbb{E}\left[f(\widetilde{L}_s,0\leq s\leq t)\right]=\mathbb{E}\left[\exp\left(-\delta\theta\int_0^t sdB_s-\delta^2\theta^2\int_0^t s^2ds\right)f(L_s,0\leq s\leq t)\right]$$
$$=\mathbb{E}\left[\exp\left(-\theta\int_0^t sdL_s-\int_0^t\Psi(\theta s)ds\right)f(L_s,0\leq s\leq t)\right].$$

The result for general γ,δ and π now follows using the independence of X and B. □

The α-stable case stated in Proposition 2.3.2 is obtained by setting $\gamma=\delta=0$, $\pi(dx)=\frac{c}{\mu}x^{-(\alpha+1)}dx$ and $\theta=1/\mu$. The Brownian case is obtained by taking $\gamma=0$, $\delta=\sqrt{\beta/\mu}$, $\theta=1/\mu$ and no Lévy measure π.

2.6.2 Size-biased reordering

In this section, we prove some elementary results about the size-biased reordering $(\hat{D}_1^n,\hat{D}_n^2,\ldots,\hat{D}_n^n)$ of the degrees. First, we prove Proposition 2.4.2.

Proof of Proposition 2.4.2. Denote the set of permutations of $\{1,2,\ldots,n\}$ by $\mathfrak{S}_n$. By definition,

$$\begin{aligned}&\mathbb{P}\left(\hat{D}_1^n=k_1,\hat{D}_2^n=k_2,\ldots,\hat{D}_n^n=k_n\right)\\&=\mathbb{P}\left(D_{\Sigma(1)}=k_1,D_{\Sigma(2)}=k_2,\ldots,D_{\Sigma(n)}=k_n\right)\\&=\sum_{\sigma\in\mathfrak{S}_n}\mathbb{P}\left(D_{\sigma(1)}=k_1,D_{\sigma(2)}=k_2,\ldots,D_{\sigma(n)}=k_n,\Sigma=\sigma\right)\\&=\sum_{\sigma\in\mathfrak{S}_n}\mathbb{P}\left(D_{\sigma(1)}=k_1,D_{\sigma(2)}=k_2,\ldots,D_{\sigma(n)}=k_n\right)\frac{k_1}{\sum_{j=1}^n k_j}\frac{k_2}{\sum_{j=2}^n k_j}\cdots\frac{k_n}{k_n}\\&=n!\,\nu_{k_1}\nu_{k_2}\ldots\nu_{k_n}\frac{k_1}{\sum_{j=1}^n k_j}\frac{k_2}{\sum_{j=2}^n k_j}\cdots\frac{k_n}{k_n},\end{aligned}$$

since $D_1, \ldots, D_n$ are i.i.d. with law ν. Rearrangement of this expression yields

$$\begin{aligned}\mathbb{P}\left(\hat{D}_1^n = k_1, \hat{D}_2^n = k_2, \ldots, \hat{D}_n^n = k_n\right) &= k_1\nu_{k_1} k_2\nu_{k_2} \ldots k_n\nu_{k_n} \prod_{i=1}^{n} \frac{(n-i+1)}{\sum_{j=i}^{n} k_j} \\ &= \frac{k_1\nu_{k_1}}{\mu}\frac{k_2\nu_{k_2}}{\mu}\cdots\frac{k_n\nu_{k_n}}{\mu}\prod_{i=1}^{n}\frac{(n-i+1)\mu}{\sum_{j=i}^{n} k_j}.\end{aligned}$$

Now

$$\begin{aligned}&\mathbb{P}\left(\hat{D}_1^n = k_1, \hat{D}_2^n = k_2, \ldots, \hat{D}_m^n = k_m\right) \\ &= \sum_{k_{m+1},\ldots,k_n \geq 1} \mathbb{P}\left(\hat{D}_1^n = k_1, \hat{D}_2^n = k_2, \ldots, \hat{D}_n^n = k_n\right) \\ &= \frac{k_1\nu_{k_1}}{\mu}\frac{k_2\nu_{k_2}}{\mu}\cdots\frac{k_m\nu_{k_m}}{\mu}\mu^m n! \sum_{k_{m+1},\ldots,k_n \geq 1} k_{m+1}\nu_{k_{m+1}}\cdots k_n\nu_{k_n}\prod_{i=1}^{n}\frac{1}{\sum_{j=i}^{n} k_j} \\ &= \frac{k_1\nu_{k_1}}{\mu}\frac{k_2\nu_{k_2}}{\mu}\cdots\frac{k_m\nu_{k_m}}{\mu}\mu^m n! \\ &\quad\times \sum_{k_{m+1},\ldots,k_n \geq 1}\prod_{i=1}^{m}\frac{1}{\sum_{j=i}^{m} k_j + \sum_{j=m+1}^{n} k_j}\nu_{k_{m+1}}\cdots\nu_{k_n}\prod_{\ell=m+1}^{n}\frac{k_\ell}{\sum_{j=\ell}^{n} k_j} \\ &= \frac{k_1\nu_{k_1}}{\mu}\frac{k_2\nu_{k_2}}{\mu}\cdots\frac{k_m\nu_{k_m}}{\mu}\mu^m \frac{n!}{(n-m)!} \\ &\quad\times \sum_{k_{m+1},\ldots,k_n \geq 1}\prod_{i=1}^{m}\frac{1}{\sum_{j=i}^{m} k_j + \sum_{j=m+1}^{n} k_j}\mathbb{P}\left(\hat{D}_1^{n-m} = k_{m+1}, \ldots, \hat{D}_{n-m}^{n-m} = k_n\right).\end{aligned}$$

We have that $\sum_{j=1}^{n-m} \hat{D}_j^{n-m} \overset{d}{=} \sum_{j=m+1}^{n} D_j = \Xi_{n-m}$. It follows that the last expression is equal to

$$\frac{k_1\nu_{k_1}}{\mu}\frac{k_2\nu_{k_2}}{\mu}\cdots\frac{k_m\nu_{k_m}}{\mu}\frac{n!\mu^m}{(n-m)!}\mathbb{E}\left[\prod_{i=1}^{m}\frac{1}{\sum_{j=i}^{m} k_j + \Xi_{n-m}}\right],$$

and the claimed result follows. □

A simple consequence of Proposition 2.4.2 is the following stochastic domination.

Lemma 2.6.3. *We have*

$$(\hat{D}_1^n, \hat{D}_2^n, \ldots, \hat{D}_n^n) \leq_{\text{st}} (Z_1, Z_2, \ldots, Z_n).$$

Proof. By Proposition 2.4.2 we have

$$\mathbb{P}\left(\hat{D}_1^n \geq d_1, \hat{D}_2^n \geq d_2, \ldots, \hat{D}_n^n \geq d_n\right) = \mathbb{E}\left[\prod_{i=1}^{n}\frac{(n-i+1)\mu}{\sum_{j=i}^{n} Z_j}\mathbb{1}_{\{Z_1 \geq d_1, Z_2 \geq d_2, \ldots, Z_n \geq d_n\}}\right].$$

Let

$$f(k_1, k_2, \ldots, k_n) = \prod_{i=1}^{n}\frac{(n-i+1)\mu}{\sum_{j=i}^{n} k_j}$$

and

$$g(k_1, k_2, \ldots, k_n) = \mathbb{1}_{\{k_1 \geq d_1, k_2 \geq d_2, \ldots, k_n \geq d_n\}}.$$

Then f is a decreasing function of its arguments and g is an increasing function of its arguments. It follows from the FKG inequality that

$$\mathbb{E}\left[f(Z_1, Z_2, \ldots, Z_n)g(Z_1, Z_2, \ldots, Z_n)\right] \leq \mathbb{E}\left[f(Z_1, Z_2, \ldots, Z_n)\right]\mathbb{E}\left[g(Z_1, Z_2, \ldots, Z_n)\right].$$

But $\mathbb{E}\left[f(Z_1, Z_2, \ldots, Z_n)\right] = 1$ and so

$$\mathbb{P}\left(\hat{D}_1^n \geq d_1, \hat{D}_2^n \geq d_2, \ldots, \hat{D}_n^n \geq d_n\right) \leq \mathbb{P}\left(Z_1 \geq d_1, Z_2 \geq d_2, \ldots, Z_n \geq d_n\right),$$

as required. □

Lemma 2.6.4. *Fix $\alpha \in (1,2)$ and suppose $m = O(n^{\beta})$ for some $\beta < \alpha/2$. Then as $n \to \infty$,*

$$\frac{1}{n}\sum_{i=1}^{m}(\hat{D}_i^n)^2 \xrightarrow{p} 0.$$

In particular, the above holds for $m = \lfloor tn^{\alpha/(\alpha+1)}\rfloor$.

Proof. By Lemma 2.6.3, it is sufficient to prove that

$$\frac{1}{n}\sum_{i=1}^{m} Z_i^2 \xrightarrow{p} 0.$$

By Theorem 2.5.9 of Durrett [71], we have

$$\limsup_{m\to\infty} \frac{1}{m^{1/\beta}}\sum_{i=1}^{m} Z_i^2 = 0 \text{ a.s.}$$

if and only if

$$\sum_{m=1}^{\infty} \mathbb{P}\left(Z_1^2 > m^{1/\beta}\right) < \infty.$$

But $\mathbb{P}\left(Z_1^2 > m^{1/\beta}\right) = \mathbb{P}\left(Z_1 > m^{1/2\beta}\right) = O(m^{-\alpha/2\beta})$, which is summable since $\alpha > 2\beta$. □

Lemma 2.6.5. *As $n \to \infty$,*

$$\frac{1}{n}\sum_{i=\lfloor tn^{\alpha/(\alpha+1)}\rfloor+1}^{n} \hat{D}_i^n \xrightarrow{p} \mu.$$

Proof. We have

$$\sum_{i=\lfloor tn^{\alpha/(\alpha+1)}\rfloor+1}^{n} \hat{D}_i^n = \sum_{i=1}^{n}\hat{D}_i^n - \sum_{i=1}^{\lfloor tn^{\alpha/(\alpha+1)}\rfloor} \hat{D}_i^n = \sum_{i=1}^{n} D_i - \sum_{i=1}^{\lfloor tn^{\alpha/(\alpha+1)}\rfloor} \hat{D}_i^n.$$

By the weak law of large numbers,

$$\frac{1}{n}\sum_{i=1}^{n} D_i \xrightarrow{p} \mathbb{E}\left[D_1\right] = \mu$$

and, since $\hat{D}_i^n \geq 1$ for $1 \leq i \leq n$, by Lemma 2.6.4 we have that

$$\frac{1}{n}\sum_{i=1}^{\lfloor tn^{\alpha/(\alpha+1)}\rfloor} \hat{D}_i^n \xrightarrow{p} 0.$$

The result follows. □

2.6.3 Convergence of the measure-change in the Brownian case

Recall that we have $\mu := \mathbb{E}[D_1]$, $\mathbb{E}\left[D_1^2\right] = 2\mu$ and $\beta := \mathbb{E}[D_1(D_1-1)(D_1-2)]$, so that $\mathbb{E}\left[D_1^3\right] = \beta + 4\mu$.

Lemma 2.6.6. *Let $\mathcal{L}(\lambda) := \mathbb{E}[\exp(-\lambda D_1)]$. Then as $\lambda \to 0$,*

$$\mathcal{L}(\lambda) = \exp\left(-\lambda\mu + \frac{\lambda^2\mu(2-\mu)}{2} - \frac{\lambda^3}{6}(\beta + 4\mu - 6\mu^2 + 2\mu^3) + o(\lambda^3)\right). \tag{2.25}$$

Proof. The first three cumulants of D_1 are

$$\mathbb{E}[D_1] = \mu, \quad \mathrm{var}(D_1) = \mu(2-\mu), \quad \mathbb{E}\left[(D_1-\mu)^3\right] = \beta + 4\mu - 6\mu^2 + 2\mu^3,$$

and the result follows immediately. □

Recall from Proposition 2.4.2 that

$$\phi^n_m(k_1, k_2, \ldots, k_m) = \mathbb{E}\left[\prod_{i=1}^m \frac{(n-i+1)\mu}{\sum_{j=i}^m k_j + \Xi_{n-m}}\right].$$

We prove the following lemma.

Lemma 2.6.7. *Let $s(0) = 0$ and $s(i) = \sum_{j=1}^i (k_j - 2)$ for $i \geq 1$. Suppose that $|s(i) - s(m)| \leq n^{1/3}\log n$ for all $0 \leq i \leq m$. Then if $m = \Theta(n^{2/3})$, we have*

$$\phi^m_n(k_1, k_2, \ldots, k_m) \geq \exp\left(\frac{1}{n\mu}\sum_{i=0}^m (s(i) - s(m)) - \frac{\beta m^3}{6\mu^3 n^2}\right)(1 + o(1)),$$

where the $o(1)$ term is independent of $k_1, \ldots, k_m \geq 1$ satisfying the conditions.

Proof. The method of proof is similar in spirit to, but somewhat more involved than, that of Lemma 2.4.7. Let us first introduce some useful notation. Let $D_i' = D_i - \mu$, the centred degree random variables, and let $\Delta_{n-m} := \Xi_{n-m} - \mu(n-m)$ be their sum. Let ψ be the log-Laplace transform of D_1',

$$\psi(\lambda) = \log \mathbb{E}\left[\exp(-\lambda D_1')\right],$$

so that as $\lambda \to 0$, we have

$$\psi(\lambda) = \frac{\lambda^2\mu(2-\mu)}{2} - \frac{\lambda^3}{6}(\beta + 4\mu - 6\mu^2 + 2\mu^3) + o(\lambda^3). \tag{2.26}$$

Now,

$$\begin{aligned}
&\phi^m_n(k_1, \ldots, k_m) \\
&= \prod_{i=1}^m \left(1 - \frac{i-1}{n}\right) \\
&\qquad \times \mathbb{E}\left[\exp\left(-\sum_{i=1}^m \log\left(1 + \frac{\Delta_{n-m} + s(m) - s(i-1) + 2(m-i+1) - \mu m}{\mu n}\right)\right)\right].
\end{aligned}$$

We use Taylor expansion in order to approximate the exponent:

$$\sum_{i=1}^{m}\left(\log\left(1-\frac{i-1}{n}\right)-\log\left(1+\frac{\Delta_{n-m}+s(m)-s(i-1)+(2-\mu)m-2(i-1)}{\mu n}\right)\right) \tag{2.27}$$

$$\begin{aligned}
&= -\frac{m^2}{2n}-\frac{m^3}{6n^2}+o(1)\\
&\quad -\frac{m\Delta_{n-m}}{n\mu}+\frac{1}{n\mu}\sum_{i=0}^{m}(s(i)-s(m))-\frac{(2-\mu)m^2}{n\mu}+\frac{m^2}{n\mu}+o(1)\\
&\quad +\frac{m(\Delta_{n-m})^2}{2n^2\mu^2}+\frac{1}{2\mu^2n^2}\sum_{i=0}^{m}(s(i)-s(m))^2+\frac{(2-\mu)^2m^3}{2\mu^2n^2}+\frac{2m^3}{3\mu^2n^2}-\frac{(2-\mu)m^3}{\mu^2n^2}\\
&\quad +o(1)\\
&\quad +\frac{\Delta_{n-m}}{\mu^2n^2}\sum_{i=0}^{m}(s(m)-s(i))-\frac{(\mu-1)m^2\Delta_{n-m}}{\mu^2n^2}+\frac{(2-\mu)m}{\mu^2n^2}\sum_{i=0}^{m}(s(m)-s(i))\\
&\quad -\frac{2}{\mu^2n^2}\sum_{i=0}^{m}i(s(m)-s(i))+\cdots
\end{aligned}$$

As Δ_{n-m} is a centred sum of i.i.d. random variables with finite variance, the central limit theorem applies and we have that $n^{-1/2}\Delta_{n-m}\xrightarrow{d}\mathrm{N}(0,\sqrt{\mu(2-\mu)})$ as $n\to\infty$. The desired lower bound will, however, be obtained by restricting to the moderate deviation event

$$\mathcal{E}_n=\left\{-(2-\mu)m-n^{7/12}\le\Delta_{n-m}\le-(2-\mu)m+n^{7/12}\right\}.$$

On this event, for any $0\le i\le m$, we have

$$|\Delta_{n-m}|=O(n^{2/3}),\quad |s(m)-s(i)|=O(n^{1/3}\log n)\quad\text{and}\quad |(2-\mu)m-2(i-1)|=O(n^{2/3}).$$

So we have

$$\begin{aligned}
&\frac{1}{2\mu^2n^2}\sum_{i=0}^{m}(s(i)-s(m))^2=o(1), &&\frac{(2-\mu)m}{\mu^2n^2}\sum_{i=0}^{m}(s(m)-s(i))=o(1),\\
&\frac{\Delta_{n-m}}{\mu^2n^2}\sum_{i=0}^{m}(s(m)-s(i))=o(1),\quad\text{and} &&-\frac{2}{\mu^2n^2}\sum_{i=0}^{m}i(s(m)-s(i))=o(1),
\end{aligned}$$

and that the remainder term (hidden in the ellipsis) in the expansion of (2.27) is $o(1)$. Using these facts we see that, on $\mathcal{E}_n$, the exponent (2.27) is equal to $F_n+o(1)$, where

$$\begin{aligned}
F_n:=\frac{1}{n\mu}\sum_{i=0}^{m}(s(i)-s(m))-\frac{(2-\mu)m^2}{2\mu n}-\frac{(2-\mu)(\mu-1)m^3}{3\mu^2n^2}\\
+\frac{m(\Delta_{n-m})^2}{2n^2\mu^2}-\left(\frac{m}{\mu n}+\frac{(\mu-1)m^2}{\mu^2n^2}\right)\Delta_{n-m}.
\end{aligned}$$

In order to find a lower bound on $\mathbb{E}\left[\exp(F_n)\mathbb{1}_{\mathcal{E}_n}\right]$, we first consider the expectation of the stochastic part,

$$\mathbb{E}\left[\exp\left(\frac{m(\Delta_{n-m})^2}{2n^2\mu^2}-\left(\frac{m}{\mu n}+\frac{(\mu-1)m^2}{\mu^2n^2}\right)\Delta_{n-m}\right)\mathbb{1}_{\mathcal{E}_n}\right].$$

Let $\theta > 0$ (we shall choose a specific value for θ shortly) and define an equivalent measure $\mathbb{Q}$ via

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(-\theta\Delta_{n-m} - (n-m)\psi(\theta)\right).$$

Because the Radon–Nikodym derivative has a product form, under $\mathbb{Q}$ the random variables $D_1', D_2', \ldots, D_{n-m}'$ are still i.i.d. and each have mean

$$\mathbb{E}_{\mathbb{Q}}\left[D_1'\right] = \frac{\mathbb{E}\left[D_1' \exp(-\theta D_1')\right]}{\mathbb{E}\left[\exp(-\theta D_1')\right]} = -\psi'(\theta)$$

and variance $\mathrm{var}_{\mathbb{Q}}(D_1') = \psi''(\theta)$. Now fix

$$\theta = \frac{m}{\mu n} + \frac{m^2}{\mu n^2},$$

so that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[\Delta_{n-m}] &= -(n-m)\psi'(\theta) \\
&= -(n-m)\left[\frac{(2-\mu)m}{n} + \frac{(2-\mu)m^2}{n^2} - \frac{(\beta+4\mu-6\mu^2+2\mu^3)m^2}{2\mu^2 n^2}\right. \\
&\qquad\qquad \left. - \frac{(\beta+4\mu-6\mu^2+2\mu^3)m^3}{2\mu^2 n^3} + o\left(\frac{m^3}{n^3}\right)\right] \\
&= -(2-\mu)m + O(n^{1/3})
\end{aligned}$$

and

$$\mathrm{var}_{\mathbb{Q}}(\Delta_{n-m}) = (n-m)\psi''(\theta) = \mu(2-\mu)(n-m) + O(n^{1/3}).$$

Using Chebyshev's inequality and the fact that $n^{1/3} \ll n^{7/12}$, it follows that

$$\mathbb{Q}(\mathcal{E}_n^c) \leq \mathbb{Q}(|\Delta_{n-m} + (2-\mu)m| > \tfrac{1}{2}n^{7/12}) \leq \frac{5\mathrm{var}_{\mathbb{Q}}(\Delta_{n-m})}{n^{7/6}} = O(n^{-1/6}).$$

So

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\frac{m(\Delta_{n-m})^2}{2n^2\mu^2} - \left(\frac{m}{\mu n} + \frac{(\mu-1)m^2}{\mu^2 n^2}\right)\Delta_{n-m}\right)\mathbb{1}_{\mathcal{E}_n}\right] \\
&\quad = \exp\left((n-m)\psi\left(\frac{m}{n\mu} + \frac{m^2}{\mu n^2}\right)\right)\mathbb{E}_{\mathbb{Q}}\left[\exp\left(\frac{m(\Delta_{n-m})^2}{2n^2\mu^2} + \frac{m^2\Delta_{n-m}}{\mu^2 n^2}\right)\mathbb{1}_{\mathcal{E}_n}\right] \\
&\quad \geq \exp\left((n-m)\psi\left(\frac{m}{n\mu} + \frac{m^2}{\mu n^2}\right)\right)\mathbb{Q}(\mathcal{E}_n) \\
&\qquad\quad \times \exp\left(\frac{m((2-\mu)m - n^{7/12})^2}{2n^2\mu^2} - \frac{m^2((2-\mu)m + n^{7/12})}{\mu^2 n^2}\right) \\
&\quad = (1+o(1))\exp\left((n-m)\psi\left(\frac{m}{n\mu} + \frac{m^2}{\mu n^2}\right) + \frac{(2-\mu)^2 m^3}{2n^2\mu^2} - \frac{(2-\mu)m^3}{\mu^2 n^2}\right).
\end{aligned}$$

Now,

$$
\begin{aligned}
&(n-m)\psi\left(\frac{m}{n\mu}+\frac{m^2}{\mu n^2}\right)\\
&= (n-m)\left[\frac{(2-\mu)m^2}{2\mu n^2}+\frac{(2-\mu)m^3}{\mu n^3}-\frac{(\beta+4\mu-6\mu^2+2\mu^3)m^3}{6\mu^3 n^3}\right]+o(1)\\
&= \frac{(2-\mu)m^2}{2\mu n}+\frac{(2-\mu)m^3}{2\mu n^2}-\frac{(\beta+4\mu-6\mu^2+2\mu^3)m^3}{6\mu^3 n^2}+o(1).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\phi_n^m(k_1,\ldots,k_m)\\
&\quad\geq (1+o(1))\mathbb{E}\left[\exp(F_n)\mathbb{1}_{\mathcal{E}_n}\right]\\
&\quad\geq (1+o(1))\exp\Bigg(\frac{(2-\mu)m^2}{2\mu n}+\frac{(2-\mu)m^3}{2\mu n^2}-\frac{(\beta+4\mu-6\mu^2+2\mu^3)m^3}{6\mu^3 n^2}\\
&\qquad\qquad +\frac{(2-\mu)^2 m^3}{2n^2\mu^2}-\frac{(2-\mu)m^3}{\mu^2 n^2}-\frac{(2-\mu)m^2}{2\mu n}-\frac{(2-\mu)(\mu-1)m^3}{3\mu^2 n^2}\\
&\qquad\qquad +\frac{1}{n\mu}\sum_{i=0}^{m}(s(i)-s(m))\Bigg)\\
&\quad= (1+o(1))\exp\left(\frac{1}{n\mu}\sum_{i=0}^{m}(s(i)-s(m))-\frac{\beta m^3}{6\mu^3 n^2}\right),
\end{aligned}
$$

as claimed. □

The event $\{|S(i)-S(m)|\leq n^{1/3}\log n \text{ for } 1\leq i\leq m\}$ has probability tending to 1 as $n\to\infty$, and so the analogue of Proposition 2.4.3 now follows exactly as in the $\alpha\in(1,2)$ case.

2.6.4 Convergence of a single large component for $\alpha\in(1,2)$

In this section, we consider a large component of the graph conditioned to have size $\lfloor xn^{\alpha/(\alpha+1)}\rfloor$, for $\alpha\in(1,2)$ only, and do the main technical work necessary to prove that it converges in distribution to a single component of the stable graph conditioned to have size x. By arguments analogous to those in Section 2.5, it is essentially sufficient to consider a single tree in the forest $\widetilde{\mathbf{F}}_n(\nu)$ of size $\lfloor xn^{\alpha/(\alpha+1)}\rfloor$, described by an excursion of the corresponding coding functions $\widetilde{S}^n$ and $\widetilde{G}^n$. The main result of this section, Theorem 2.6.8, is a conditioned version of Theorem 2.4.1, which says that these excursions converge jointly in distribution to normalised excursions of $\widetilde{L}$ and $\widetilde{H}$ of length x. (This is precisely the analogue of Theorem 2.2.5 in the measure-changed setting.) At the end of the section, we sketch how to obtain the metric space scaling limit of a single large component of the graph.

For simplicity, we will make the assumption that the support of the law of D_1 is $\mathbb{Z}_+$ so that excursions of any strictly positive length occur with positive probability. This assumption is not necessary, since the condition $\mathbb{P}(D_1=k)\sim ck^{-(\alpha+2)}$ implies that the greatest common divisor of $\{k\geq 2:\mathbb{P}(D_1-1=k)>0\}$ is 1, so that the claimed results all hold for n sufficiently large.

Recall that $(S(k), k \geq 0)$ is a random walk which is skip-free to the left and in the domain of attraction of an α-stable Lévy process. Let

$$\mathcal{E}_m = \Big\{ S(k) \geq 0 \text{ for } 0 < k < m,\, S(m) = -1 \Big\},$$

the event that the first m steps form an excursion above the running minimum. If $m = \lfloor n^{\alpha/(\alpha+1)} x \rfloor$ then, by Theorem 2.2.5,

$$\mathbb{E}\left[f\left(n^{-1/(\alpha+1)} S(\lfloor n^{\alpha/(\alpha+1)} t \rfloor), n^{-(\alpha-1)/(\alpha+1)} G(\lfloor n^{\alpha/(\alpha+1)} t \rfloor), 0 \leq t \leq x \right) \Big| \mathcal{E}_m \right] \\ \to \mathbb{N}^{(x)} \left[f(\mathbb{e}, \mathbb{h}) \right].$$

More generally, write

$$\mathcal{E}_{m_1,m_2} = \Big\{ S(m_1) = \min_{0 \leq k \leq m_2 - 1} S(k) = S(m_2) + 1 \Big\}$$

(so that $\mathcal{E}_m = \mathcal{E}_{0,m}$) and, similarly,

$$\widetilde{\mathcal{E}}^n_{m_1,m_2} = \Big\{ \widetilde{S}^n(m_1) = \min_{0 \leq k \leq m_2 - 1} \widetilde{S}^n(k) = \widetilde{S}^n(m_2) + 1 \Big\},$$

so that $\widetilde{\mathcal{E}}^n_{m_1,m_2}$ is the event that there is an excursion of $\widetilde{S}^n$ above its running minimum between times m_1 and m_2 (recall that $\widetilde{S}^n$ and $\widetilde{G}^n$ have the same excursion intervals). This, of course, corresponds to a component of size $m_2 - m_1$. Observe that the corresponding excursion of the height process starts and ends at 0. We will prove the following result.

Theorem 2.6.8. *For any bounded continuous test function f, $0 \leq t_1 < t_2$ such that $t_2 - t_1 = x$, and $m_1 = \lfloor t_1 n^{\alpha/(\alpha+1)} \rfloor$, $m_2 = \lfloor t_2 n^{\alpha/(\alpha+1)} \rfloor$, $m = m_2 - m_1$ then*

$$\mathbb{E}\Big[f\Big(n^{-1/(\alpha+1)} [\widetilde{S}^n(\lfloor (t_1 + t) n^{\alpha/(\alpha+1)} \rfloor) - \widetilde{S}^n(\lfloor t_1 n^{\alpha/(\alpha+1)} \rfloor)], \\ n^{-(\alpha-1)/(\alpha+1)} \widetilde{G}^n(\lfloor (t_1 + t) n^{\alpha/(\alpha+1)} \rfloor), \quad 0 \leq t \leq n^{-\alpha/(\alpha+1)} m \Big) \Big| \widetilde{\mathcal{E}}^n_{m_1,m_2} \Big] \\ \to \frac{\mathbb{N}^{(x)} \left[\exp\left(\frac{1}{\mu} \int_0^x \mathbb{e}(t) dt \right) f(\mathbb{e}, \mathbb{h}) \right]}{\mathbb{N}^{(x)} \left[\exp\left(\frac{1}{\mu} \int_0^x \mathbb{e}(t) dt \right) \right]}$$

as $n \to \infty$.

We need to prove a refinement of Lemma 2.4.7, to show that the change of measure is well-behaved at times when the process attains a new minimum.

Proposition 2.6.9. *Fix $T > 0$. For $n \geq 1$ and $m \leq T n^{\frac{\alpha}{\alpha+1}}$, let $k_1^{(n)}, k_2^{(n)}, \ldots, k_m^{(n)} \geq 1$ and let $s^{(n)}(i) = \sum_{j=1}^i (k_j^{(n)} - 2)$ be such that $s^{(n)}(0) = 0$ and $s^{(n)}(i) > s^{(n)}(m)$ for $1 \leq i \leq m-1$. Then*

$$\phi_n^m(k_1^{(n)}, k_2^{(n)}, \ldots, k_m^{(n)}) = (1 + \delta_n) \exp\left(\frac{1}{n\mu} \sum_{i=0}^m (s^{(n)}(i) - s^{(n)}(m)) - \frac{C_\alpha m^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1} n^\alpha} \right),$$

where δ_n depends only on n and T, and $\delta_n \to 0$ as $n \to \infty$.

We start with a technical lemma.

Lemma 2.6.10. *Suppose that* $m = O(n^{\alpha/(\alpha+1)})$. *Let* $E_1, E_2, \ldots$ *be i.i.d. standard exponential random variables. Suppose that for each* n *we have a sequence* $a_1^{(n)}, a_2^{(n)}, \ldots, a_m^{(n)}$ *such that* $a_i^{(n)} \in (0, Km/n)$ *for all* $1 \leq i \leq m$ *for some constant* K.

(a) We have

$$\sum_{i=1}^{m} a_i^{(n)}(E_i - 1) \to 0$$

in L^2.

(b) For any $p > 1$, *there exists a constant* $C > 0$ *such that*

$$\mathbb{E}\left[\exp\left(p\sum_{i=1}^{m} a_i^{(n)}(E_i - 1)\right)\right] \leq C\exp\left(\frac{2pK^2m^3}{n^2}\right).$$

Both the convergence in (*a*) *and the bound in* (*b*) *are uniform in sequences* $(a_i^{(n)})$ *satisfying the above conditions.*

Proof. (a) Since the sum is centred, we have

$$\mathbb{E}\left[\left(\sum_{i=1}^{m} a_i^{(n)}(E_i - 1)\right)^2\right] = \operatorname{var}\left(\sum_{i=1}^{m} a_i^{(n)}(E_i - 1)\right) = \sum_{i=1}^{m}(a_i^{(n)})^2 \leq \frac{K^2m^3}{n^2} \to 0$$

as $n \to \infty$.

(b) For $0 < a < 1/2$ we have

$$\mathbb{E}\left[\exp(aE_1 - a)\right] = \frac{e^{-a}}{1-a} = e^{-a}\left(1 + \frac{a}{1-a}\right) \leq \exp\left(-a + \frac{a}{1-a}\right) \leq \exp(2a^2).$$

So for sufficiently large n we have

$$\mathbb{E}\left[\exp\left(p\sum_{i=1}^{m} a_i^{(n)}(E_i - 1)\right)\right] \leq \exp\left(2p\sum_{i=1}^{m}(a_i^{(n)})^2\right) \leq \exp\left(\frac{2pK^2m^3}{n^2}\right). \qquad \square$$

Proof of Proposition 2.6.9. The lower bound does not rely on $s^{(n)}$ attaining a new minimum at time m, and has already been proved in Lemma 2.4.7; we need a matching upper bound. To ease readability, we will suppress the superscripts on $k_i^{(n)}$ and $s^{(n)}(i)$. Now,

$$\begin{aligned}
&\mathbb{E}\left[\prod_{i=1}^{m} \frac{(n-i+1)\mu}{\sum_{j=i}^{m} k_j + \Xi_{n-m}}\right] \\
&= \prod_{i=1}^{m-1}\left(1 - \frac{i}{n}\right)\mathbb{E}\left[\prod_{i=1}^{m} \frac{n\mu}{s(m) - s(i-1) + 2(m-i+1) + \Xi_{n-m}}\right] \\
&= \prod_{i=1}^{m-1}\left(1 - \frac{i}{n}\right)\mathbb{E}\left[\exp\left(-\sum_{i=1}^{m}\left\{\frac{(s(m) - s(i-1) + 2(m-i+1) + \Xi_{n-m}}{n\mu} - 1\right\}E_i\right)\right],
\end{aligned}$$

where $E_1, E_2, \dots$ are i.i.d. standard exponential random variables, independent of Ξ_{n-m}. We shall first consider the expectation conditionally on $E_1, E_2, \dots, E_m$. Write A_m for the quantity

$$\prod_{i=1}^{m-1}\left(1-\frac{i}{n}\right)$$
$$\times \mathbb{E}\left[\exp\left(-\sum_{i=1}^{m}\left\{\frac{(s(m)-s(i-1)+2(m-i+1)+\Xi_{n-m}}{n\mu}-1\right\}E_i\right)\middle| E_1,\dots,E_m\right].$$

Let $C>0$ be a constant to be chosen later. We will split $\mathbb{E}[A_m]$ into two parts, so that

$$\mathbb{E}\left[\prod_{i=1}^{m}\frac{(n-i+1)\mu}{\sum_{j=i}^{m}k_j+\Xi_{n-m}}\right]=\mathbb{E}\left[A_m\mathbb{1}_{\{\sum_{i=1}^m E_i>Cn^{\alpha/(\alpha+1)}\}}\right]+\mathbb{E}\left[A_m\mathbb{1}_{\{\sum_{i=1}^m E_i\le Cn^{\alpha/(\alpha+1)}\}}\right].$$

We deal with the first term on the right-hand side first. Since $k_j \ge 1$ for all j and $\Xi_{n-m} \ge n-m$ a.s., we have the crude bound

$$s(m)-s(i-1)+2(m-i+1)+\Xi_{n-m}\ge n-i+1>n-m>n/2$$

for $1 \le i \le m$ and all n sufficiently large that $m/n < 1/2$. Then

$$\begin{aligned}
\mathbb{E}\left[A_m\mathbb{1}_{\{\sum_{i=1}^m E_i>Cn^{\alpha/(\alpha+1)}\}}\right] &\le \mathbb{E}\left[\exp\left(\left(1-\frac{1}{2\mu}\right)\sum_{i=1}^m E_i\right)\mathbb{1}_{\{\sum_{i=1}^m E_i>Cn^{\alpha/(\alpha+1)}\}}\right]\\
&=\int_{Cn^{\alpha/(\alpha+1)}}^{\infty}\exp\left(x-\frac{x}{2\mu}\right)\frac{e^{-x}x^{m-1}}{\Gamma(m)}dx\\
&=\int_{Cn^{\alpha/(\alpha+1)}}^{\infty}\exp\left(-\frac{x}{2\mu}\right)\frac{x^{m-1}}{\Gamma(m)}dx\\
&=(2\mu)^m\mathbb{P}\left(\frac{1}{2\mu}\sum_{i=1}^m E_i>Cn^{\alpha/(\alpha+1)}\right).
\end{aligned}$$

By Markov's inequality, this last quantity is bounded above by

$$(2\mu)^m\mathbb{E}\left[\exp\left(\frac{1}{2\mu}\sum_{i=1}^m E_i\right)\right]\exp\left(-Cn^{\alpha/(\alpha+1)}\right)=\left(\frac{(2\mu)^2}{2\mu-1}\right)^m\exp(-Cn^{\alpha/(\alpha+1)})\to 0$$

as $n \to \infty$, as long as we take $C > T(2\log(2\mu)-\log(2\mu-1))$, which we henceforth assume.

Let us now turn to the expectation of A_m on the event $\{\sum_{i=1}^m E_i \le Cn^{\alpha/(\alpha+1)}\}$. Since

$$\prod_{i=1}^{m-1}\left(1-\frac{i}{n}\right)\le\exp\left(-\frac{m(m-1)}{2n}\right),$$

we have

$$\begin{aligned}
&A_m\mathbb{1}_{\{\sum_{i=1}^m E_i\le Cn^{\alpha/(\alpha+1)}\}}\\
&\quad\le\exp\left(-\frac{m(m-1)}{2n}+\frac{1}{n\mu}\sum_{i=1}^m(s(i-1)-s(m))E_i-\sum_{i=1}^m\left(\frac{2(m-i+1)}{n\mu}-1\right)E_i\right)\\
&\qquad\times\mathbb{E}\left[\exp\left(-\left(\frac{1}{n\mu}\sum_{i=1}^m E_i\right)\Xi_{n-m}\right)\middle| E_1,\dots,E_m\right]\mathbb{1}_{\{\sum_{i=1}^m E_i\le Cn^{\alpha/(\alpha+1)}\}}.
\end{aligned}$$

On the event $\left\{\sum_{i=1}^m E_i \le Cn^{\alpha/(\alpha+1)}\right\}$, we have $\frac{1}{n\mu}\sum_{i=1}^m E_i = o(1)$. Hence, we may apply the asymptotic formula (2.16) for the Laplace transform of D_1 to obtain that on the event $\left\{\sum_{i=1}^m E_i \le Cn^{\alpha/(\alpha+1)}\right\}$ we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-\left(\frac{1}{n\mu}\sum_{i=1}^m E_i\right)\Xi_{n-m}\right)\middle| E_1,\dots,E_m\right] \\
&= \exp\left(-\frac{(n-m)}{n}\sum_{i=1}^m E_i + \frac{(2-\mu)}{2\mu n}\left(\sum_{i=1}^m E_i\right)^2 - \frac{C_\alpha}{(\alpha+1)n^\alpha\mu^{\alpha+1}}\left(\sum_{i=1}^m E_i\right)^{\alpha+1} + o(1)\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
&A_m \mathbb{1}_{\{\sum_{i=1}^m E_i \le Cn^{\alpha/(\alpha+1)}\}} \\
&\le \exp\left(\frac{1}{n\mu}\sum_{i=1}^m (s(i-1)-s(m))E_i - \frac{C_\alpha}{(\alpha+1)n^\alpha\mu^{\alpha+1}}\left(\sum_{i=1}^m E_i\right)^{\alpha+1} + o(1)\right) \\
&\quad\times \exp\left(\frac{(2-\mu)}{2\mu n}\left(\sum_{i=1}^m E_i\right)^2 - \frac{2}{n\mu}\sum_{i=1}^m (m-i+1)E_i + \frac{m}{n}\sum_{i=1}^m E_i - \frac{m^2}{2n}\right) \\
&\quad\times \mathbb{1}_{\{\sum_{i=1}^m E_i \le Cn^{\alpha/(\alpha+1)}\}}.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\frac{(2-\mu)}{2\mu n}\left(\sum_{i=1}^m E_i\right)^2 - \frac{2}{n\mu}\sum_{i=1}^m (m-i+1)E_i + \frac{m}{n}\sum_{i=1}^m E_i - \frac{m^2}{2n} \\
&= \frac{(2-\mu)}{2\mu n}\left(m + \sum_{i=1}^m E_i\right)\sum_{i=1}^m (E_i - 1) - \frac{2}{n\mu}\sum_{i=1}^m (m-i+1)(E_i-1) \\
&\quad + \frac{m}{n}\sum_{i=1}^m (E_i-1) + \frac{m}{n\mu}.
\end{aligned}$$

So

$$\begin{aligned}
&A_m \mathbb{1}_{\{\sum_{i=1}^m E_i \le Cn^{\alpha/(\alpha+1)}\}} \\
&\le \exp\left(\frac{1}{n\mu}\sum_{i=0}^m (s(i)-s(m)) - \frac{C_\alpha m^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}n^\alpha} + o(1)\right)\exp(\chi_n)\mathbb{1}_{\{\sum_{i=1}^m E_i \le Cn^{\alpha/(\alpha+1)}\}}, \qquad (2.28)
\end{aligned}$$

where

$$\begin{aligned}
\chi_n &= \frac{1}{n\mu}\sum_{i=1}^m (s(i-1)-s(m))(E_i-1) - \frac{2}{n\mu}\sum_{i=1}^m (m-i+1)(E_i-1) + \frac{m}{n}\sum_{i=1}^m (E_i-1) \\
&\quad + \frac{(2-\mu)}{2\mu n}\left(m + \sum_{i=1}^m E_i\right)\sum_{i=1}^m (E_i-1) - \frac{C_\alpha m^{\alpha+1}}{(\alpha+1)n^\alpha\mu^{\alpha+1}}\left(\left(\frac{1}{m}\sum_{i=1}^m E_i\right)^{\alpha+1} - 1\right).
\end{aligned}$$

We need to understand the asymptotics of the expectation of the right-hand side of (2.28). Recall that s has steps down of magnitude at most 1, so that we have the crude bound $s(i)-$

$s(m) \leq m - i + 1 \leq m$ for all $0 \leq i \leq m$. So by Lemma 2.6.10(a), we get

$$\frac{1}{n\mu}\sum_{i=1}^{m}(s(i-1)-s(m))(E_i-1) - \frac{2}{n\mu}\sum_{i=1}^{m}(m-i+1)(E_i-1) + \frac{m}{n}\sum_{i=1}^{m}(E_i-1) \to 0$$

in L^2, as $n \to \infty$. We have

$$\mathbb{E}\left[\left(\frac{(2-\mu)}{2\mu n}\left(m + \sum_{i=1}^{m} E_i\right)\sum_{i=1}^{m}(E_i-1)\right)^2 \mathbb{1}_{\{\sum_{i=1}^{m} E_i \leq Cn^{\alpha/(\alpha+1)}\}}\right]$$
$$\leq \frac{(2-\mu)^2(T+C)^2 n^{2\alpha/(\alpha+1)}}{4\mu^2 n^2}\mathbb{E}\left[\left(\sum_{j=1}^{m}(E_j-1)\right)^2\right] \leq \frac{(T+C)^2 T n^{3\alpha/(\alpha+1)}}{\mu^2 n^2} \to 0,$$

since $n^{\alpha/(\alpha+1)}/n^2 = n^{(\alpha-2)/(\alpha-1)} = o(1)$. If $m \to \infty$ as $n \to \infty$, it follows straightforwardly from the weak law of large numbers that

$$\left(\frac{1}{m}\sum_{i=1}^{m} E_i\right)^{\alpha+1} \xrightarrow{p} 1$$

and so, for any $m \leq Tn^{\alpha/(\alpha+1)}$, we have

$$\frac{C_\alpha m^{\alpha+1}}{(\alpha+1)n^\alpha \mu^{\alpha+1}}\left(\left(\frac{1}{m}\sum_{i=1}^{m} E_i\right)^{\alpha+1} - 1\right) \xrightarrow{p} 0.$$

These results imply that $\chi_n \xrightarrow{p} 0$ on $\{\sum_{i=1}^{m} E_i \leq Cn^{\alpha/(\alpha+1)}\}$, as $n \to \infty$ and so

$$\exp(\chi_n)\mathbb{1}_{\{\sum_{i=1}^{m} E_i \leq Cn^{\alpha/(\alpha+1)}\}} \xrightarrow{p} 1.$$

It remains to show that

$$\mathbb{E}\left[\exp(\chi_n)\mathbb{1}_{\{\sum_{i=1}^{m} E_i \leq Cn^{\alpha/(\alpha+1)}\}}\right] \to 1$$

as $n \to \infty$, for which we require uniform integrability. Now, we have

$$\exp\left(\frac{(2-\mu)}{2\mu n}\left(m + \sum_{i=1}^{m} E_i\right)\sum_{i=1}^{m}(E_i-1)\right)\mathbb{1}_{\{\sum_{i=1}^{m} E_i \leq Cn^{\alpha/(\alpha+1)}\}}$$
$$\leq 1 + \exp\left(\frac{(2-\mu)}{2\mu n}(C+1)m\sum_{i=1}^{m}(E_i-1)\right).$$

Hence,

$$\exp(\chi_n)\mathbb{1}_{\{\sum_{i=1}^{m} E_i \leq Cn^{\alpha/(\alpha+1)}\}}$$
$$\leq \exp\left(\frac{C_\alpha m^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}n^\alpha}\right)$$
$$\times\left[\exp\left(\sum_{i=1}^{m}\left\{\frac{(s(i-1)-s(m))}{n\mu} - \frac{2(m-i+1)}{n\mu} + \frac{m}{n}\right\}(E_i-1)\right)\right.$$
$$\left. + \exp\left(\sum_{i=1}^{m}\left\{\frac{(s(i-1)-s(m))}{n\mu} - \frac{2(m-i+1)}{n\mu} + \frac{m}{n} + \frac{(2-\mu)(C+1)m}{2\mu n}\right\}(E_i-1)\right)\right].$$

Applying Lemma 2.6.10(b), we see that both terms are bounded in L^p for $p > 1$. Hence, the sequence $(\exp(\chi_n)\mathbb{1}_{\{\sum_{i=1}^m E_i \le Cn^{\alpha/(\alpha+1)}\}},\ n \ge 1)$ is uniformly integrable and we may deduce that

$$\mathbb{E}\left[\exp(\chi_n)\mathbb{1}_{\{\sum_{i=1}^m E_i \le Cn^{\alpha/(\alpha+1)}\}}\right] \to 1,$$

which concludes the proof. $\square$

We will also need the following lemma.

Lemma 2.6.11. *Fix $\theta > 0$ and let $m = \lfloor Tn^{\alpha/(\alpha+1)} \rfloor$. Then we have the following uniform integrability: for $K > 0$,*

$$\limsup_{K\to\infty} \sup_{n\ge1} \mathbb{E}\left[\exp\left(\frac{\theta}{n}\sum_{i=0}^{m}(S(i)-S(m))\right)\mathbb{1}_{\{\frac{1}{n}\sum_{i=0}^m (S(i)-S(m))>K\}}\middle|\mathcal{E}_m\right] = 0.$$

Proof. The proof uses similar ingredients to the proof of Lemma 2.3.6. On the event $\mathcal{E}_m$ we have

$$\frac{\theta}{n}\sum_{i=0}^{m}(S(i)-S(m)) \le \frac{\theta m}{n}\left(1+\max_{0\le i\le m} S(i)\right) \le \theta T^{(\alpha+1)/\alpha} m^{-1/\alpha}\left(1+\max_{0\le i\le m} S(i)\right).$$

So it will be sufficient to show that we have

$$\limsup_{K\to\infty} \sup_{m\ge1} \mathbb{E}\left[\exp\left(\theta m^{-1/\alpha}\max_{0\le i\le m} S(i)\right)\mathbb{1}_{\{m^{-1/\alpha}\max_{0\le i\le m} S(i)>K\}}\middle|\mathcal{E}_m\right] = 0.$$

We have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\theta m^{-1/\alpha}\max_{0\le i\le m} S(i)\right)\mathbb{1}_{\{m^{-1/\alpha}\max_{0\le i\le m} S(i)>K\}}\middle|\mathcal{E}_m\right] \\
&= \sum_{k=\lfloor Km^{1/\alpha}\rfloor+1}^{\infty} e^{\theta m^{-1/\alpha}k}\,\mathbb{P}\left(\max_{0\le i\le m} S(i) = k\middle|\mathcal{E}_m\right) \\
&\le e^{(K+1)\theta}\,\mathbb{P}\left(m^{-1/\alpha}\max_{0\le i\le m} S(i) > K\middle|\mathcal{E}_m\right) \\
&\quad + \sum_{k=\lfloor Km^{1/\alpha}\rfloor+2}^{\infty} \theta m^{-1/\alpha} e^{\theta m^{-1/\alpha}k}\,\mathbb{P}\left(\max_{0\le i\le m} S(i) \ge k\middle|\mathcal{E}_m\right),
\end{aligned}$$

by summation by parts and the fact that $e^{\theta m^{-1/\alpha}k} - e^{\theta m^{-1/\alpha}(k-1)} \le m^{-1/\alpha}\theta e^{\theta m^{-1/\alpha}k}$. Theorem 9 of Kortchemski [92] gives that for any $\delta \in (0, \alpha/(\alpha-1))$, there exist universal constants $C_1, C_2 > 0$ such that

$$\mathbb{P}\left(m^{-1/\alpha}\max_{0\le i\le m} S(i) \ge u\middle|\mathcal{E}_m\right) \le C_1\exp(-C_2 u^{\delta}).$$

We take $\delta \in (1, \alpha/(\alpha-1))$. So then

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\theta m^{-1/\alpha} \max_{0\le i\le m} S(i)\right) \mathbb{1}_{\{m^{-1/\alpha}\max_{0\le i\le m} S(i)>K\}} \middle| \mathcal{E}_m\right] \\
&\quad \le e^{(K+1)\theta}\mathbb{P}\left(m^{-1/\alpha}\max_{0\le i\le m} S(i) > K \middle| \mathcal{E}_m\right) \\
&\qquad + \int_K^\infty \theta e^{\theta x}\mathbb{P}\left(m^{-1/\alpha}\max_{0\le i\le m} S(i) \ge x-1 \middle| \mathcal{E}_m\right) dx \\
&\quad \le C_1\exp((K+1)\theta - C_2K^\delta) + \int_K^\infty C_1\theta\exp(\theta x - C_2(x-1)^\delta)dx,
\end{aligned}$$

which clearly tends to 0 as $K \to \infty$ since $\delta > 1$. The result follows. □

Proof of Theorem 2.6.8. Recall that $S(k) = \sum_{i=1}^k (Z_i - 2)$, where $Z_1, Z_2, \ldots$ are i.i.d. with the size-biased degree distribution. Then we have

$$\begin{aligned}
&\mathbb{E}\left[f\left(n^{-\frac{1}{\alpha+1}}[\widetilde{S}^n(\lfloor (t_1+\cdot)n^{\frac{\alpha}{\alpha+1}}\rfloor) - \widetilde{S}^n(\lfloor t_1 n^{\frac{\alpha}{\alpha+1}}\rfloor)], \widetilde{G}^n(\lfloor (t_1+\cdot)n^{\frac{\alpha}{\alpha+1}}\rfloor)\right) \middle| \widetilde{\mathcal{E}}^n_{m_1,m_2}\right] \\
&= \frac{\mathbb{E}\left[f\left(n^{-\frac{1}{\alpha+1}}[\widetilde{S}^n(\lfloor (t_1+\cdot)n^{\frac{\alpha}{\alpha+1}}\rfloor) - \widetilde{S}^n(\lfloor t_1 n^{\frac{\alpha}{\alpha+1}}\rfloor)], n^{-\frac{\alpha-1}{\alpha+1}}\widetilde{G}^n(\lfloor (t_1+\cdot)n^{\frac{\alpha}{\alpha+1}}\rfloor)\right)\mathbb{1}_{\widetilde{\mathcal{E}}^n_{m_1,m_2}}\right]}{\mathbb{E}\left[\mathbb{1}_{\widetilde{\mathcal{E}}^n_{m_1,m_2}}\right]}.
\end{aligned}$$

Using the change of measure, this is equal to

$$\begin{aligned}
&\frac{\mathbb{E}\left[\Phi(n,\lfloor t_2 n^{\frac{\alpha}{\alpha+1}}\rfloor) f\left(n^{-\frac{1}{\alpha+1}}[S(\lfloor (t_1+\cdot)n^{\frac{\alpha}{\alpha+1}}\rfloor) - S(\lfloor t_1 n^{\frac{\alpha}{\alpha+1}}\rfloor)], n^{-\frac{\alpha-1}{\alpha+1}} G(\lfloor (t_1+\cdot)n^{\frac{\alpha}{\alpha+1}}\rfloor)\right)\mathbb{1}_{\mathcal{E}_{m_1,m_2}}\right]}{\mathbb{E}\left[\Phi(n,\lfloor t_2 n^{\frac{\alpha}{\alpha+1}}\rfloor)\mathbb{1}_{\mathcal{E}_{m_1,m_2}}\right]} \\
&= \frac{\mathbb{E}\left[\Phi(n,\lfloor t_2 n^{\frac{\alpha}{\alpha+1}}\rfloor) f\left(n^{-\frac{1}{\alpha+1}}[S(\lfloor (t_1+\cdot)n^{\frac{\alpha}{\alpha+1}}\rfloor) - S(\lfloor t_1 n^{\frac{\alpha}{\alpha+1}}\rfloor)], n^{-\frac{\alpha-1}{\alpha+1}} G(\lfloor (t_1+\cdot)n^{\frac{\alpha}{\alpha+1}}\rfloor)\right)\middle|\mathcal{E}_{m_1,m_2}\right]}{\mathbb{E}\left[\Phi(n,\lfloor t_2 n^{\frac{\alpha}{\alpha+1}}\rfloor)\middle|\mathcal{E}_{m_1,m_2}\right]}. \qquad (2.29)
\end{aligned}$$

By Proposition 2.6.9, we have that on the event $\mathcal{E}_{m_1,m_2}$,

$$\Phi(n,\lfloor t_2 n^{\frac{\alpha}{\alpha+1}}\rfloor) = (1+o(1))\exp\left(\frac{1}{\mu n}\sum_{i=0}^{m_2}(S(i) - S(m_2)) - \frac{C_\alpha t_2^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right), \qquad (2.30)$$

where the $o(1)$ is uniform on $\mathcal{E}_{m_1,m_2}$. But using that $S(m_2) = S(m_1) - 1$, we get

$$\begin{aligned}
\Phi(n,\lfloor t_2 n^{\frac{\alpha}{\alpha+1}}\rfloor) &= (1+o(1))\exp\left(\frac{1}{\mu n}\sum_{i=0}^{m_1-1}(S(i) - S(m_1)) + \frac{m_1}{\mu n} - \frac{C_\alpha t_2^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right) \\
&\qquad \times \exp\left(\frac{1}{\mu n}\sum_{i=m_1}^{m_2}([S(i) - S(m_1)] - [S(m_2) - S(m_1)])\right).
\end{aligned}$$

The increments of the random walk S are independent, and so the first and second terms in this product are independent. The first term is also independent of the argument of the function f. So in both the numerator and denominator of the fraction (2.29), we may cancel a factor of

$$\mathbb{E}\left[\exp\left(\frac{1}{\mu n}\sum_{i=0}^{m_1-1}(S(i) - S(m_1)) + \frac{m_1}{\mu n} - \frac{C_\alpha t_2^{\alpha+1}}{(\alpha+1)\mu^{\alpha+1}}\right)\middle|\mathcal{E}_{m_1,m_2}\right].$$

Using also the stationarity of the increments of S and the fact that $m = m_2 - m_1$, we then obtain that (2.29) is equal to $(1 + o(1))$ times

$$\frac{\mathbb{E}\Big[\exp\Big(\frac{1}{\mu n}\sum_{i=0}^{m}(S(i)-S(m))\Big) f\Big(n^{-\frac{1}{\alpha+1}} S(\lfloor tn^{\frac{\alpha}{\alpha+1}}\rfloor), n^{-\frac{\alpha-1}{\alpha+1}} G(\lfloor tn^{\frac{\alpha}{\alpha+1}}\rfloor), 0 \le t \le n^{-\frac{\alpha}{\alpha+1}} m\Big)\Big|\mathcal{E}_m\Big]}{\mathbb{E}\Big[\exp\Big(\frac{1}{\mu n}\sum_{i=0}^{m}(S(i)-S(m))\Big)\Big|\mathcal{E}_m\Big]}.$$

Lemma 2.6.11 gives us the requisite uniform integrability in order to now deduce the result from Theorem 2.2.5 and the continuous mapping theorem. □

Let us briefly sketch how this result gives a scaling limit for a single component of $\mathbf{M}_n(\nu)$ or $\mathbf{G}_n(\nu)$ conditioned to have size $\lfloor xn^{\alpha/(\alpha+1)}\rfloor$. First note that for any $\epsilon > 0$ there exists a time $T > 0$ such that any component of size $\lfloor xn^{\alpha/(\alpha+1)}\rfloor$ is discovered before time $\lfloor Tn^{\alpha/(\alpha+1)}\rfloor$ with probability exceeding $1 - \epsilon$, uniformly in n sufficiently large. Any such component discovered before time $\lfloor Tn^{\alpha/(\alpha+1)}\rfloor$ corresponds to a tree of size $\approx xn^{\alpha/(\alpha+1)}$ in the forest encoded by $\widetilde{S}^n$ and $\widetilde{G}^n$ and, indeed, this tree is asymptotically indistinguishable in the Gromov–Hausdorff–Prokhorov sense from a spanning tree of the graph component. The locations of the back-edges can then be handled in exactly the same way as in the unconditioned setting.

Chapter 3

Gaussian Free Field level-set percolation on regular random graphs

This chapter stems from the preprint [56], that has not yet been submitted.

Abstract. In this chapter, we study the level-set of the zero-average Gaussian Free Field on a uniform random d-regular graph above an arbitrary level $h \in (-\infty, h_\star)$, where $h_\star$ is the level-set percolation threshold of the GFF on the d-regular tree $\mathbb{T}_d$. We prove that w.h.p as the number n of vertices diverges, the GFF has a unique giant connected component $\mathcal{C}_1^{(n)}$ of size $\eta(h)n + o(n)$, where $\eta(h)$ is the probability that the root percolates in the corresponding GFF level-set on $\mathbb{T}_d$. This gives a positive answer to the conjecture of [4] for most regular graphs. We also prove that the second largest component has size $\Theta(\log n)$.
Moreover, we show that $\mathcal{C}_1^{(n)}$ shares the following similarities with the giant component of the supercritical Erdős-Rényi random graph. First, the diameter and the typical distance between vertices are $\Theta(\log n)$. Second, the 2-core and the kernel encompass a given positive proportion of the vertices. Third, the local structure is a branching process conditioned to survive, namely the level-set percolation cluster of the root in $\mathbb{T}_d$ (in the Erdős-Rényi case, it is known to be a Galton-Watson tree with a Poisson distribution for the offspring).

3.1 Introduction

3.1.1 Overview

The Gaussian Free Field (GFF) on a transient graph $\mathcal{G}$ is a Gaussian process indexed by the vertices. Its covariance is given by the Green function, hence the GFF carries a lot of information on the structure of $\mathcal{G}$ and on the behaviour of random walks, giving a base motivation for

its study.
Level-set percolation of the GFF has been investigated since the 1980s ([48, 111]). Lately, one important incentive has been to gain information on the vacant set of random interlacements ([104, 131]), via Dynkin-type isomorphism theorems ([73, 125]). It was subject to much attention in the last decade on $\mathbb{Z}^d$ ([65, 104, 121, 130]). On such a lattice where the Green function decays polynomially with the distance between vertices, it provides a percolation model with long-range interactions.
More recently, level-set percolation was studied on transient rooted trees ([3, 2, 132]). There is a phase transition at a critical threshold $h_\star \in \mathbb{R}$: if $h < h_\star$, the connected component of the root in the level-set above h of the GFF has a positive probability to be infinite, and if $h > h_\star$, this probability is zero.
One can define an analogous field on a finite connected graph, the **zero-average Gaussian Free Field**, whose covariance is given by the **zero-average Green function** (see Section 3.1.2). A natural question is whether some characteristics of the GFF on an infinite graph $\mathcal{G}$ can be transferred to a sequence of finite graphs $(\mathcal{G}_n)_{n\geq 0}$ whose local limit is $\mathcal{G}$. For instance, one might ask whether a phase transition for the existence of an infinite connected component of the level-set in $\mathcal{G}$ corresponds to a phase transition for the emergence of a "macroscopic" component of size $\Theta(|\mathcal{G}_n|)$ in the level set in $\mathcal{G}_n$. For $\mathcal{G} = \mathbb{Z}^d$, Abächerli [1] studied the zero-average GFF on the torus.
Abächerli and Černý recently investigated the GFF on the d-regular tree $\mathbb{T}_d$ [3], and the zero-average GFF on some d-regular graphs (large girth expanders) in a companion paper [4]. In this setting, many essential questions (such as the value of $h_\star$, or the sharpness of the phase transition at $h_\star$ for the zero-average GFF) remain open. In this paper, we answer some of them, and relate the percolation level-sets to other classical random graphs, in particular the Erdős-Rényi model (Section 3.1.4).
To do so, we refine some properties of [3] on $\mathbb{T}_d$. In a work in progress [57], we study further the GFF on $\mathbb{T}_d$ and the random walk on the level-sets.

3.1.2 Setting

In all this work, we fix an integer $d \geq 3$. We denote $\mathbb{T}_d$ the infinite d-regular tree rooted at an arbitrary vertex $\circ$, and $\mathcal{G}_n$ a uniform d-regular random graph for $n \geq 1$ (if d is odd, consider only even n). Let V_n be its vertex set and π_n be the uniform measure on V_n, i.e. $\pi_n(x) = 1/n$ for every $x \in V_n$.

Gaussian Free Field on regular trees

The GFF $\varphi_{\mathbb{T}_d}$ on $\mathbb{T}_d$ is a centred Gaussian field $(\varphi_{\mathbb{T}_d}(x))_{x\in\mathbb{T}_d}$ indexed by the vertices of $\mathbb{T}_d$, and with covariances given by the Green function $G_{\mathbb{T}_d}$: for all vertices $x, y \in \mathbb{T}_d$, we set

$\mathrm{Cov}(\varphi_{\mathbb{T}_d}(x), \varphi_{\mathbb{T}_d}(y)) = G_{\mathbb{T}_d}(x, y)$. Recall that

$$G_{\mathbb{T}_d}(x, y) = \mathbf{E}_x^{\mathbb{T}_d}\left[\sum_{k\geq 0} \mathbf{1}_{\{X_k = y\}}\right]$$

where $(X_k)_{k\geq 0}$ is a discrete-time SRW (Simple Random Walk) on $\mathbb{T}_d$. In general, we will denote $\mathbf{P}_\mu^{\mathcal{G}}$ the law of a SRW on a graph $\mathcal{G}$ with initial distribution μ.

Gaussian Free Field on finite graphs

If $\mathcal{G}_n$ is connected, the zero-average GFF $\psi_{\mathcal{G}_n}$ on $\mathcal{G}_n$ is a centred Gaussian field $(\psi_{\mathcal{G}_n}(x))_{x\in\mathcal{G}_n}$ indexed by the vertices of $\mathcal{G}_n$, and with covariances given by the zero-average Green function $G_{\mathcal{G}_n}$ on $\mathcal{G}_n$: for all $x, y \in \mathcal{G}_n$, we set

$$\mathrm{Cov}(\psi_{\mathcal{G}_n}(x), \psi_{\mathcal{G}_n}(y)) = G_{\mathcal{G}_n}(x, y) := \mathbf{E}_x^{\mathcal{G}_n}\left[\int_0^{+\infty}\left(\mathbf{1}_{\{\overline{X}_t = y\}} - \frac{1}{n}\right) dt\right]$$

where $(\overline{X}_t)_{t\geq 0}$ is a continuous time SRW on $\mathcal{G}_n$ started at x with Exp(1) independent jump-times. Precisely, let $(\zeta_i)_{i\geq 1}$ be a sequence of independent exponential variables of parameter 1. Let $(X_k)_{k\geq 0}$ be a SRW started at x, independent of $(\xi_i)_{i\geq 1}$. Then for all $t \geq 0$, we define $\overline{X}_t := X_{k(t)}$, with $k(t) := \sup_{k\geq 0} \sum_{i=1}^k \zeta_i \leq t$.
The function $G_{\mathcal{G}_n}$ is symmetric, finite and positive semidefinite. This ensures that $\psi_{\mathcal{G}_n}$ is well-defined (see [1] for details, in particular Remark 1.2).

Two layers of randomness

Denote $\mathbb{P}_{ann}$ and $\mathbb{E}_{ann}$ the annealed law and expectation for the joint realization of $\mathcal{G}_n$ and of $\psi_{\mathcal{G}_n}$ on it. For a fixed realization of $\mathcal{G}_n$, denote $\mathbb{P}^{\mathcal{G}_n}$ and $\mathbb{E}^{\mathcal{G}_n}$ the quenched law and expectation.

3.1.3 Results

Define the level set $E_{\varphi_{\mathbb{T}_d}}^{\geq h} := \{x \in \mathbb{T}_d \,|\, \varphi_{\mathbb{T}_d}(x) \geq h\}$. Let $\mathcal{C}_\circ^h$ be the connected component of $E_{\varphi_{\mathbb{T}_d}}^{\geq h}$ containing the root $\circ$. Similarly, define the level sets $E_{\psi_{\mathcal{G}_n}}^{\geq h} := \{x \in \mathcal{G}_n \,|\, \psi_{\mathcal{G}_n}(x) \geq h\}$ for $n \geq 1$. For $i \geq 1$, let $\mathcal{C}_i^{(n)}$ be the i-th largest connected component of $E_{\psi_{\mathcal{G}_n}}^{\geq h}$. In [132], Sznitman showed that there exists a constant $h_\star > 0$ such that

$$\text{if } h > h_\star,\ \eta(h) := \mathbb{P}^{\mathbb{T}_d}(|\mathcal{C}_\circ^h| = +\infty) = 0, \text{ and if } h < h_\star,\ \eta(h) > 0. \tag{3.1}$$

In [3] (Theorems 4.3 and 5.1), Abächerli and Černý showed that if $h > h_\star$, the size of $\mathcal{C}_\circ^h$ has exponential moments, and if $h < h_\star$, $\mathcal{C}_\circ^h$ has a positive probability to grow exponentially. In [4] (Theorems 3.1 and 4.1), they proved that $\mathbb{P}_{ann}$-w.h.p.: if $h > h_\star$, $|\mathcal{C}_1^{(n)}| = O(\log n)$, and if $h < h_\star$, at least ξn vertices of $E_{\psi_{\mathcal{G}_n}}^{\geq h}$ are in components of size at least n^δ, for some constants $\delta, \xi > 0$ depending on h. They even found deterministic conditions on $\mathcal{G}_n$, satisfied w.h.p., so that these events hold $\mathbb{P}^{\mathcal{G}_n}$-w.h.p. (see the discussion in Section 3.1.4).

Thus, in the supercritical case $h < h_\star$, a positive proportion of the vertices is in at least "mesoscopic" components (there is no explicit lower bound for δ).
This paper focuses exclusively on the supercritical case. We prove the existence of a giant component:

Theorem 3.1.1. *Let $h < h_\star$. It holds:*

$$\frac{|\mathcal{C}_1^{(n)}|}{n} \xrightarrow{\mathbb{P}_{ann}} \eta(h), \tag{3.2}$$

where $\xrightarrow{\mathbb{P}_{ann}}$ stands for convergence in $\mathbb{P}_{ann}$-probability as $n \to +\infty$. Moreover, there exists $K_0 > 0$ such that

$$\mathbb{P}_{ann}\left(K_0^{-1}\log n \le |\mathcal{C}_2^{(n)}| \le K_0 \log n\right) \underset{n\to+\infty}{\longrightarrow} 1. \tag{3.3}$$

Note that by Markov's inequality, for any $\varepsilon > 0$ and any sequence of events $(\mathcal{E}_n)_{n\ge1}$ such that $\mathbb{P}_{ann}(\mathcal{E}_n) \to 1$, w.h.p. $\mathcal{G}_n$ is such that $\mathbb{P}^{\mathcal{G}_n}(\mathcal{E}_n) \ge 1-\varepsilon$. Thus, w.h.p. on $\mathcal{G}_n$, the conclusions of Theorem 3.1.1 hold with arbitrarily large $\mathbb{P}^{\mathcal{G}_n}$-probability.

We also establish some structural properties of $\mathcal{C}_1^{(n)}$. Let $\mathbf{C}^{(n)}$ be the **2-core** of $\mathcal{C}_1^{(n)}$, obtained by deleting recursively the vertices of degree 1 of $\mathcal{C}_1^{(n)}$ and their edges. Let $\mathbf{K}^{(n)}$ be the **kernel** of $\mathcal{C}_1^{(n)}$, i.e. $\mathbf{C}^{(n)}$ where simple paths are contracted to a single edge, so that the vertices of $\mathbf{K}^{(n)}$ are those of $\mathbf{C}^{(n)}$ with degree at least 3.

Theorem 3.1.2. Global structure of $\mathcal{C}_1^{(n)}$
Fix $h < h_\star$. There exist $K_1, K_2 > 0$ such that

$$\frac{|\mathbf{C}^{(n)}|}{n} \xrightarrow{\mathbb{P}_{ann}} K_1 \tag{3.4}$$

and

$$\frac{|\mathbf{K}^{(n)}|}{n} \xrightarrow{\mathbb{P}_{ann}} K_2. \tag{3.5}$$

Moreover, there exists $K_3 > 0$ such that if $D_1^{(n)}$ is the diameter of $\mathcal{C}_1^{(n)}$, then

$$\mathbb{P}_{ann}(D_1^{(n)} \le K_3 \log n) \underset{n\to+\infty}{\longrightarrow} 1. \tag{3.6}$$

Last, there exists $\lambda_h > 1$ such that for every $\varepsilon > 0$,

$$\pi_{2,n}(\{(x,y) \in (\mathcal{C}_1^{(n)})^2,\ (1-\varepsilon)\log_{\lambda_h} n \le d_{\mathcal{C}_1^{[n)}}(x,y) \le (1+\varepsilon)\log_{\lambda_h} n\}) \xrightarrow{\mathbb{P}_{ann}} 1, \tag{3.7}$$

where $\pi_{2,n}$ is the uniform measure on $(\mathcal{C}_1^{(n)})^2$ and $d_{\mathcal{C}_1^{(n)}}$ the usual graph distance on $\mathcal{C}_1^{(n)}$. In other words, the typical distance between vertices of $\mathcal{C}_1^{(n)}$ is $\log_{\lambda_h} n$.

We will see in Section 3.3 that λ_h is the growth rate of $\mathcal{C}_\circ^h$ conditioned on being infinite.
Say that a random graph G is the **local limit** of the random graph sequence $(G_n)_{n\ge1}$ if G_n converges to G in distribution w.r.t to the local topology (see for instance the lecture notes of Curien [59] for a precise definition). We prove that the local limit of $\mathcal{C}_1^{(n)}$ is $\mathcal{C}_\circ^h$ conditioned to be infinite.

Theorem 3.1.3. Local limit of $\mathcal{C}_1^{(n)}$
For every radius $k \geq 1$, for every rooted tree T of height k, let $V_n^{(T)} := \{x \in \mathcal{C}_1^{(n)}, B_{\mathcal{C}_1^{(n)}}(x,k) = T\}$ and $p_T := \mathbb{P}^{\mathbb{T}_d}(B_{\mathcal{C}_\circ^h}(\circ, k) = T \mid |\mathcal{C}_\circ^h| = +\infty)$. Then

$$\frac{|V_n^{(T)}|}{|\mathcal{C}_1^{(n)}|} \xrightarrow{\mathbb{P}_{ann}} p_T.$$

3.1.4 Discussion and open questions

GFF percolation versus bond percolation
The graph $E_{\psi_{\mathcal{G}_n}}^{\geq h}$ undergoes the same phase transition as some classical bond percolation models for the size of the largest connected component. We draw a comparison with the Erdős-Rényi random graph (i.e. bond percolation on the complete graph), introduced by Gilbert in [79]: for a constant $c > 0$ and $n \in \mathbb{N}$, $\mathrm{ER}(n, c/n)$ is the graph on n vertices such that for every pair of vertices x, y, there is an edge between x and y with probability c/n, independently of all other pairs of vertices. Erdős and Rényi [74] showed that the supercritical regime corresponds to $c > 1$ and the subcritical regime to $c < 1$. Theorems 3.1.1, 3.1.2 and 3.1.3 hold for $\mathrm{ER}(n, c/n)$ as $n \to +\infty$, for any fixed $c > 1$, the tree $\mathcal{C}_\circ^h$ being replaced by a Galton-Watson tree whose offspring distribution is Poisson with parameter c, and λ_h being replaced by c.
As for Bernoulli bond percolation on $\mathcal{G}_n$ (each edge of $\mathcal{G}_n$ is deleted with probability $1-p$, independently of the others), the same phase transition holds for the size of the largest connected component, the critical threshold being $p = 1/(d-1)$ (Theorem 3.2 of [18]).

The structure of $\mathcal{C}_1^{(n)}$
It was shown recently in [64] that the distribution of the giant component of $\mathrm{ER}(n, p/n)$ is continuous w.r.t. to a random graph which can be explicitly described. Its kernel is a configuration model whose vertices have i.i.d. degrees with a Poisson distribution (conditioned on being at least 3). In particular, it is an expander. The lengths of the simple paths in the 2-core are i.i.d. geometric random variables. See Theorem 1 of [64] for details. This implies a result analogous to Theorem 3.1.2 for $\mathrm{ER}(n, p/n)$.
We conjecture that the kernel $\mathbf{K}^{(n)}$ is an expander for every $h < h_\star$. The main obstacle to gathering information on its global structure is that if $\psi_{\mathcal{G}_n}$ is revealed on a positive proportion of the vertices of $\mathbf{K}^{(n)}$ (and hence of $\mathcal{G}_n$), then it could affect substantially $\psi_{\mathcal{G}_n}$ on the remaining vertices. In particular, if $h > 0$ is large enough, we could imagine that the average of $\psi_{\mathcal{G}_n}$ on the discovered vertices is positive. But by (3.18), the average of the GFF on the remaining vertices would be negative, hence below the threshold h.

Deterministic regular graphs
The results of [4] and [18] hold in fact for any deterministic sequence of large-girth expanders (conditions (I) and (II) in Proposition 3.2.1), which is w.h.p. the case for $\mathcal{G}_n$. Very recently,

after a first preprint of our work, Černý [136] gave another proof of (3.2) that holds under these deterministic conditions. He also showed that $|\mathcal{C}_2^{(n)}| = o(n)$ w.h.p. His approach is very different, and uses notably a novel decomposition of the GFF as an infinite sum of fields with finite range interactions, introduced in [68] and [67].
In our proofs, averaging on the randomness of $\mathcal{G}_n$ is a crucial ingredient to control the presence of cycles on large subgraphs of $\mathcal{G}_n$, and allows us to extend some arguments of [4], where $\psi_{\mathcal{G}_n}$ is locally approximated by $\varphi_{\mathbb{T}_d}$.
We conjecture that those deterministic conditions are not sufficient for (3.3) to hold. This was shown for the Bernoulli bond percolation in [93] (Theorem 2): for every $a \in (0,1)$, one can build a sequence $(G_n)_{n\geq 1}$ satisfying (I) and (II) such that the second largest connected component has at least n^a vertices (the second largest component first grows exponentially on a tree-like ball until it has a polynomial size, and then is "trapped" in zones where the expansion of the graph is close to an arbitrarily small constant).

Behaviour at criticality
Almost nothing is known about $\mathcal{C}_\circ^{h_\star}$ (even the value of $h_\star$), though one might conjecture that it is a.s. finite, as critical Galton-Watson trees. In the Erdős-Rényi model, the critical case is by far the most interesting. When $p = 1 + \Theta(n^{-1/3})$, the i-th largest connected component has size $\varsigma_i n^{2/3}$, where ς_i is an a.s. finite random variable (see the celebrated paper by Aldous [13]). Its structure is similar to a modified Brownian tree [6]. Hence, there is a finite but arbitrary large number of components of the biggest order, and some of their main characteristics, such as their cardinality, are random.
It would be interesting to look for the size and shape of $|\mathcal{C}_i^{(n)}|$ for $h = h_\star$, or for an hypothetical critical window $(h_\star - \varepsilon_n, h_\star + \varepsilon_n)$ for a sequence $(\varepsilon_n)_{n\geq 1}$ converging to 0 at an appropriate speed.

3.1.5 Proof outline

Our proofs rely on two main arguments:

1) An annealed exploration of $E^{\geq h}_{\psi_{\mathcal{G}_n}}$ (Proposition 3.2.4), where the structure of $\mathcal{G}_n$ is progressively revealed (there is a standard sequential construction of $\mathcal{G}_n$, see Section 3.2). Each newly discovered vertex is given an independent standard normal variable. Then $\psi_{\mathcal{G}_n}$ is built via a recursive procedure, using these Gaussian variables (Proposition 3.2.3).

2) A comparison of $\psi_{\mathcal{G}_n}$ and $\varphi_{\mathbb{T}_d}$ (Proposition 3.4.1) : on a tree-like subgraph T of $\mathcal{G}_n$, such that there are no cycles in $\mathcal{G}_n$ at distance $\kappa \log\log n$ of T for a large enough constant κ, there is a bijective map Φ between T and an isomorphic subtree of $\mathbb{T}_d$ and a coupling of $\psi_{\mathcal{G}_n}$ and $\varphi_{\mathbb{T}_d}$ so that
$$\sup_{y\in T} |\psi_{\mathcal{G}_n}(y) - \varphi_{\mathbb{T}_d}(\Phi(y))| \leq \log^{-1} n.$$

We stress the fact that we reveal $\psi_{\mathcal{G}_n}$ only *after* having explored $\mathcal{G}_n$: if we reveal $\psi_{\mathcal{G}_n}$ at a given vertex, it conditions the structure of $\mathcal{G}_n$ and thus the pairings of the still unmatched half-edges, so that we cannot use the sequential construction any more to further explore the graph. Hence, during the exploration, we will need to build an approximate version of $\psi_{\mathcal{G}_n}$, depending on the Gaussian variables of 1). This makes some proofs tedious, in particular that of (3.3).

The base exploration

The exploration that we will perform in all proofs, with some modifications, is as follows: pick $x \in V_n$, and reveal its connected component $\mathcal{C}_x^{\mathcal{G}_n,h}$ in $E_{\psi_{\mathcal{G}_n}}^{\geq h}$ in a breadth-first way, as well as its neighbourhood up to distance $a_n = \kappa \log\log n$. Until we meet a cycle, the explored zone is a tree T_x, growing at least like $\mathcal{C}_\circ^{h+\log^{-1} n}$, and at most like $\mathcal{C}_\circ^{h-\log^{-1} n}$ by 2).
On one hand, $\mathcal{C}_\circ^{h+\log^{-1} n}$ has a probability $\simeq \eta(h + \log^{-1} n) = \eta(h) + o(1)$ to be infinite, with a growth rate $\lambda_h > 1$ (Section 3.3.2). On the other hand, the probability to create a cycle is $o(1)$ as long as we reveal $o(\sqrt{n})$ vertices (since we perform $o(\sqrt{n})$ pairings of half-edges having each a probability $o(\sqrt{n})/n$ to involve two already discovered vertices). Thus, with $\mathbb{P}_{ann}$-probability $\eta(h) + o(1)$, T_x and ∂T_x will reach a size $\Theta(\sqrt{n} \log^{-\kappa'} n)$ for some constant $\kappa' > 0$ (Proposition 3.5.1).
Conversely, $\mathcal{C}_\circ^{h-\log^{-1} n}$ has a probability $1 - \eta(h - \log^{-1} n) = 1 - \eta(h) + o(1)$ to be finite, and with $\mathbb{P}_{ann}$-probability $1 - \eta(h) + o(1)$, $|\mathcal{C}_x^{\mathcal{G}_n,h}| = o(\sqrt{n})$ (Proposition 3.5.4).

Proof of (3.2).

First, we show that for any two vertices $x, y \in V_n$, there is a $\mathbb{P}_{ann}$-probability $\eta(h)^2 + o(1)$ that they are connected in $E_{\psi_{\mathcal{G}_n}}^{\geq h}$. To do so, we explore $\mathcal{C}_x^{\mathcal{G}_n,h}$ and $\mathcal{C}_y^{\mathcal{G}_n,h}$, that we couple with independent copies of $\mathcal{C}_\circ^{h+\log^{-1} n}$, so that with probability $\eta(h)^2 + o(1)$, ∂T_x and ∂T_y have $\Theta(\sqrt{n} \log^{-\kappa'} n)$ vertices. The explorations from x and y are disjoint with probability $1 - o(1)$, since $o(\sqrt{n})$ vertices have been explored. Then, we draw multiple paths between T_x and T_y (with an "envelope" of radius $\Theta(\log\log n)$ around each of them to allow the use of the approximation 2)), the **joining balls** (Section 3.6.1). The probability that $E_{\psi_{\mathcal{G}_n}}^{\geq h}$ percolates through at least one of these paths is $1 - o(1)$.
Second, we prove by a second moment argument that $\mathbb{P}_{ann}$-w.h.p., the number of couples $(x, y) \in V_n^2$ such that $y \in \mathcal{C}_x^{\mathcal{G}_n,h}$ is $(\eta(h)^2 + o(1))n^2$ (Lemma 3.6.4).
Third, knowing that $|\mathcal{C}_x^{\mathcal{G}_n,h}| = o(\sqrt{n})$ with $\mathbb{P}_{ann}$-probability $1 - \eta(h) + o(1)$, we deduce in the same way that at least $(1 - \eta(h) + o(1))n$ vertices are in connected components of size $o(\sqrt{n})$ (Lemma 3.6.3).
Those two facts together force the existence of a connected component of size $(\eta(h) + o(1))n$.

Proof of (3.3).

The most difficult part is the upper bound. We show that for K_0 large enough, for $x \in V_n$,

$\mathbb{P}_{ann}(K_0 \log n \leq |\mathcal{C}_x^{\mathcal{G}_n,h}| \leq K_0^{-1} n) = o(1/n)$, and conclude by a union bound on x and a corollary of the proof of (3.2), namely that $|\mathcal{C}_2^{(n)}|/n \overset{\mathbb{P}_{ann}}{\longrightarrow} 0$.
The greater precision $o(1/n)$ requires three additional ingredients:

- the size of $\mathcal{C}_\circ^h$ conditioned on being finite has exponential moments (Proposition 3.3.6), in particular, $\mathbb{P}^{\mathbb{T}_d}(|\mathcal{C}_\circ^h| \geq c \log n,\ |\mathcal{C}_\circ^h| < +\infty) = o(1/n)$ for a large enough constant c;
- when exploring k vertices around x, there is a probability $\Theta(k^2/n)$ that a cycle arises, so that we will need to handle at least one cycle to fully explore $\mathcal{C}_x^{\mathcal{G}_n,h}$;
- we need a better approximation of $\psi_{\mathcal{G}_n}$ than $\log^{-1} n$ in 2): with probability at least $\Theta(1/n)$, we will meet too many vertices with an approximate value of $\psi_{\mathcal{G}_n}$ that are in $[h - \log^{-1} n, h + \log^{-1} n]$, so that we can not tell whether they are in $\mathcal{C}_x^{\mathcal{G}_n,h}$ or not before the end of the exploration. To remedy this, we replace the "security radius" a_n in 2) by some $r_n = \Theta(\log n)$, so that we approximate $\psi_{\mathcal{G}_n}$ up to a difference $n^{-\Theta(1)}$.

Other proofs.
The proofs of Theorems 3.1.2 and 3.1.3 are based on slightly modified explorations, and are much simpler.

3.1.6 Plan of the rest of the paper

In Section 3.2, we review some basic properties of $\mathcal{G}_n$ (structure, Green function and GFF). In Section 3.3, we study the GFF on $\mathbb{T}_d$. In Section 3.4, we establish a coupling between recursive constructions of the GFF on $\mathbb{T}_d$ and on a tree-like neighbourhood of $\mathcal{G}_n$. In Section 3.5, we explore the connected component of a vertex in $E_{\psi_{\mathcal{G}_n}}^{\geq h}$. In Section 3.6, we prove (3.2). In Section 3.7, we prove (3.3). In Section 3.8, we prove Theorems 3.1.2 and 3.1.3.

3.1.7 Further definitions

In this paper, edges are non-oriented, and thus graphs are undirected. For any graph G, denote d_G the standard graph distance on its vertex set V, and for every vertex x and $R \geq 0$, let $B_G(x,R) := \{y,\ d_G(x,y) \leq R\}$ and $\partial B_G(x,R+1) = B_G(x,R+1) \setminus B_G(x,R)$. For any $S \subseteq V$, let similarly $B_G(S,R) := \cup_{x \in S} B_G(x,R)$ and $\partial B_G(S,R+1) = B_G(S,R+1) \setminus B_G(S,R)$. If A is a subgraph of G with vertex set S, denote $B_G(A,R) = B_G(S,R)$. If x and y are neighbours, we denote $B_G(x,y,R)$ the subgraph of G obtained by taking all paths of length R starting at x and not going through y.
The **tree excess** of a finite graph G is $\text{tx}(G) = e - v + 1$, where $v := |V|$ and e is the number of edges in G.An important remark (in particular for Proposition 3.4.1) is that for any subgraph A of G and $R \in \mathbb{N}$, $\text{tx}(B_G(A,R)) \geq \text{tx}(A)$, with equality if and only if $B_G(A,R)$ has the same number of cycles and the same number of connected components as A. Note also that if G is connected, $\text{tx}(G) = 0$ if and only if G is a tree, i.e. has no cycle.

A **rooted tree** is a tree T with a distinguished vertex $\circ$, the **root**. The **height** $\mathfrak{h}_T(x)$ of a vertex x in T is $d_T(\circ, x)$. If T is finite, its **boundary** ∂T is the set of vertices of maximal height. The **subtree from** x is the subtree made of the vertices y such that x is on any path from $\circ$ to y. The **offspring** of x is the set of vertices of its subtree. For $r \geq 0$, the r**-offspring** of x is its offspring at distance r of x, and its **offspring up to generation** r is its offspring at distance at most r. If y is in the 1-offspring of x, then y is a **child** of x, and x is its **parent**. In this case, write $x = \overline{y}$.

If x, y are neighbours in T, the **cone from** x **out of** y is the rooted subtree of T with root x and vertex set $\{z \in T \,|\, y \text{ is not on the shortest path from } x \text{ to } z\}$.

An **isomorphism** between two rooted trees T and T' is a bijection $\Phi : T \to T'$ preserving the root and the height, and such that for all vertices $x, y \in T$, there is an edge between x and y if and only if there is an edge between $\Phi(x)$ and $\Phi(y)$.

Unless mention of the contrary, all random walks are in discrete time. We will write T_A (resp. H_A) for the first exit (resp. hitting) time of A by a SRW.

For two probability distributions μ, μ' on $\mathbb{R}$, we write $\mu \underset{st.}{\leq} \mu'$ (or $\mu' \underset{st.}{\geq} \mu$) if μ' dominates stochastically μ, i.e. there exist two random variables $X \sim \mu$ and $X' \sim \mu'$ on the same probability space such that $X \leq X'$ a.s.

3.2 Basic properties of $\mathcal{G}_n$

3.2.1 Structure and Green function

The graph $\mathcal{G}_n$ can be generated sequentially as follows: attach d half-edges to each vertex of V_n. Pick an arbitrary half-edge, and match it to another half-edge chosen uniformly at random. Choose a remaining half-edge and match it to another unpaired half-edge chosen uniformly and independently of the previous matching, and so on until all half-edges have been paired. The resulting multi-graph $\mathcal{M}_n$ is not necessary **simple**, i.e. it might have loops and multiple edges. The probability that $\mathcal{M}_n$ is simple has a positive limit as $n \to +\infty$, and conditionally on $\{\mathcal{M}_n \text{ is simple}\}$, $\mathcal{M}_n$ is distributed as $\mathcal{G}_n$ (see for instance Section 7 of [135], in particular Proposition 7.13 for a reference).

In particular, an event true w.h.p. on $\mathcal{M}_n$ is also true w.h.p. on $\mathcal{G}_n$, so that it is enough to prove all our results on $\mathcal{M}_n$. In the rest of the paper, we will even write $\mathcal{G}_n$ for $\mathcal{M}_n$ for the sake of simplicity.

This Section is devoted to proving this result:

Proposition 3.2.1. *There exists $K_3 > 0$ such that w.h.p. as $n \to +\infty$, $\mathcal{G}_n$ satisfies:*

*(I) $\mathcal{G}_n$ is a K_3-**expander**, i.e. the spectral gap $\lambda_{\mathcal{G}_n}$ of $\mathcal{G}_n$ is at least K_3 (the spectral gap is the smallest eigenvalue of $I - P$ where I is the identity matrix and P the transition matrix of the SRW on $\mathcal{G}_n$),*

(II) for all $x \in \mathcal{G}_n$, $B_{\mathcal{G}_n}(x, \lfloor K_3 \log n \rfloor)$ contains at most one cycle.

Moreover, there exists $K_4 > 0$ such that w.h.p. on $\mathcal{G}_n$, it holds: for all $x \in V_n$ such that $\mathrm{tx}(B_{\mathcal{G}_n}(x, \lfloor K_4 \log \log n \rfloor)) = 0$,

$$\left| G_{\mathcal{G}_n}(x,x) - \frac{d-1}{d-2} \right| \leq \log^{-6} n. \tag{3.8}$$

If moreover y is a neighbour of x,

$$\left| G_{\mathcal{G}_n}(x,y) - \frac{1}{d-2} \right| \leq \log^{-6} n. \tag{3.9}$$

Say that a given realization of $\mathcal{G}_n$ is a **good graph** when (I), (II), (3.8) and (3.9) hold. The equations (3.8) and (3.9) illustrate the fact that $G_{\mathcal{G}_n}$ is close to $G_{\mathbb{T}_d}$ on a tree-like neighbourhood: it is well-known that for all $x, y \in \mathbb{T}_d$,

$$G_{\mathbb{T}_d}(x,y) = \frac{(d-1)^{1-d_{\mathbb{T}_d}(x,y)}}{d-2}. \tag{3.10}$$

A quick computation can be found in [138], Lemma 1.24.

By Proposition 1.1 of [4], (I) and (II) imply that for some $K_5, K_6 > 0$ and for n large enough, for all $x, y \in V_n$,

$$|G_{\mathcal{G}_n}(x,y)| \leq \frac{K_5}{(d-1)^{d_{\mathcal{G}_n}(x,y)}} \vee n^{-K_6}. \tag{3.11}$$

Throughout this paper, we will often make binomial estimations, because the number of edges between two sets of vertices in $\mathcal{G}_n$ is close to a binomial random variable, as highlighted in the Lemma below. We will use repeatedly the following classical inequalities: for $n \geq m \geq 0$ and $p \in (0,1)$, if $Z \sim \mathrm{Bin}(n,p)$, one has

$$\mathbb{P}(Z \geq m) \leq \binom{n}{m} p^m, \ \mathbb{P}(Z \leq m) \leq \binom{n}{m}(1-p)^m, \ \binom{n}{m} \leq \frac{n^m}{m!} \leq n^m. \tag{3.12}$$

The following Lemma is an important consequence of the sequential construction of $\mathcal{G}_n$.

Lemma 3.2.2 (Binomial number of connections). *Let $m \in \mathbb{N}$, let W_0, W_1 be disjoint subsets of V_n. Write $m_0 := |W_0|$ and $m_1 := |W_1|$. Suppose the only information we have on $\mathcal{G}_n$ is a set E of its edges that has been revealed. Let $m_E := |E|$ and denote $\mathbb{P}^E$ the law of $\mathcal{G}_n$ conditionally on this information. Repeat the following operation m times: pick an arbitrary vertex $v \in W_0$ having at least one unmatched half-edge, and pair it with an other half-edge. Add its other endpoint v' in W_0, if it was not already in it. Let s be the number of times that $v' \in W_1$. Suppose that $m_E + m + m_1 < n$. Then*

$$s \underset{st.}{\leq} \mathrm{Bin}\left(m, \frac{m_1}{n - (m_E + m)}\right), \tag{3.13}$$

In particular,

a) for any fixed $k \in \mathbb{N}$, there exists $C(k) > 0$ so that for n large enough, if $m_E + m < n/2$,

$$\mathbb{P}^E(s \geq k) \leq C(k) \left(\frac{m_1 m}{n}\right)^k. \tag{3.14}$$

b) for $k = k(n) \to +\infty$ and n large enough, if we have $m_E + m < n/2$ and $kn > 6(m_1 + m_0 + m_E)m$, then

$$\mathbb{P}^E(s \geq k) \leq 0.99^k. \tag{3.15}$$

Proof. Pick $v \in W_0$, such that v has an unmatched half-edge e. There are at most m_1 vertices in W_1, so that there are at most dm_1 unmatched half-edges that belong to its vertices. And the total number of unmatched half-edges is at least $dn - 2(|E| + m) \geq d(n - m_E - m)$. Thus, the probability that e is matched with a half-edge belonging to a vertex of W_1 is not greater than $\frac{dm_1}{d(n-m_E-m)} = \frac{m_1}{n-(m_E+m)}$. The successive matchings are performed independently, and (3.13) follows.

Let $Z \sim Bin\left(m, \frac{m_1}{n-(m_E+m)}\right)$. By (3.12), for $k \in \mathbb{N}$, we have

$$\mathbb{P}^E(Z \geq k) \leq \binom{m}{k} \left(\frac{m_1}{n-(m_E+m)}\right)^k \leq \binom{m}{k} \left(\frac{m_1}{n/2}\right)^k \leq \frac{2^k}{k!} \frac{m_1^k m^k}{n^k}.$$

This yields (3.14). Moreover, if $k \to +\infty$ as $n \to +\infty$ and $kn > 6(m_1 + m_0 + m_E)m$, by Stirling's formula, we have that for n large enough: $\mathbb{P}^E(Z \geq k) \leq \left(\frac{(2e+0.1)m_1 m}{kn}\right)^k < 0.99^k$, and (3.15) follows. □

It is straightforward to adapt this when s counts the number of times that v' was in W_0 (and there is no set W_1). m_1 is replaced by $m_0 + m$ in (3.13) and (3.14), and (3.15) does not change. Throughout this paper, we will refer to these equations without mentioning explicitly if we count the connections from W_0 to W_1 or from W_0 to itself.

Proof of Proposition 3.2.1. By Theorem 1 of [44] and the Cheeger bound, $\mathcal{G}_n$ satisfies (I) w.h.p. As for (II), fix $K_3 > 0$. For all $x \in V_n$, one obtains $B_{\mathcal{G}_n}(x, \lfloor K_3 \log n \rfloor)$ by proceeding to at most $d(d-1)^{\lfloor K_3 \log n \rfloor}$ pairings of half-edges. If K_3 is small enough, $d(d-1)^{\lfloor K_3 \log n \rfloor} < n^{1/5} - 1$ so that by (3.14) with $m_0 = 1$, $m_E = 0$, $m = d(d-1)^{\lfloor K_3 \log n \rfloor}$ and $k = 2$, for n large enough:

$$\mathbb{P}(\text{tx}(B_{\mathcal{G}_n}(x, \lfloor K_3 \log n \rfloor)) \geq 2) \leq C(2) \left(\frac{n^{2/5}}{n}\right)^2 \leq n^{-11/10}.$$

By a union bound on $x \in V_n$, w.h.p. $\mathcal{G}_n$ is such that for all $x \in V_n$, $\text{tx}(B_{\mathcal{G}_n}(x, \lfloor K_3 \log n \rfloor)) \leq 1$. We now establish (3.8). Note that $U := B_{\mathcal{G}_n}(x, \lfloor K_4 \log\log n \rfloor)$ and $W := B_{\mathbb{T}_d}(\circ, \lfloor K_4 \log\log n \rfloor)$ are isomorphic. Then $G^U_{\mathcal{G}_n}(x, x) = G^W_{\mathbb{T}_d}(\circ, \circ)$, where we let

$$G^A_{\mathcal{G}_n}(y, z) := \mathbf{E}^{\mathcal{G}_n}_y [\textstyle\sum_{k=0}^{T_A} \mathbf{1}_{\{X_k = z\}}] \text{ for every } y, z \in V_n \text{ and } A \subsetneq V_n.$$

Recall that T_A is the exit time of A by the SRW $(X_k)_{k \geq 0}$. Similarly for every $B \subsetneq \mathbb{T}_d$ and $y, z \in \mathbb{T}_d$, we define

$$G^B_{\mathbb{T}_d}(y, z) := \mathbf{E}^{\mathbb{T}_d}_y [\sum_{k=0}^{T_B} \mathbf{1}_{\{X_k = z\}}]. \tag{3.16}$$

On one hand, by the strong Markov property applied to the exit time T_W,

$$G^W_{\mathbb{T}_d}(\circ, \circ) = G_{\mathbb{T}_d}(\circ, \circ) - \mathbf{E}^{\mathbb{T}_d}_\circ [G_{\mathbb{T}_d}(\circ, X_{T_W})] = \frac{d-1}{d-2} - \mathbf{E}^{\mathbb{T}_d}_\circ [G_{\mathbb{T}_d}(\circ, X_{T_W})].$$

On the other hand, by Lemma 1.4 of [1], for all $y,z\in V_n$ and $A\subsetneq V_n$:

$$G^A_{\mathcal{G}_n}(y,z)=G_{\mathcal{G}_n}(y,z)-\mathbf{E}^{\mathcal{G}_n}_y[G_{\mathcal{G}_n}(z,X_{T_A})]+\frac{\mathbf{E}^{\mathcal{G}_n}_y[T_A]}{n}, \tag{3.17}$$

so that $G^U_{\mathcal{G}_n}(x,x)=G_{\mathcal{G}_n}(x,x)-\mathbf{E}^{\mathcal{G}_n}_x[G_{\mathcal{G}_n}(x,X_{T_U})]+\frac{\mathbf{E}^{\mathcal{G}_n}_x[T_U]}{n}$. Therefore,

$$\left|G_{\mathcal{G}_n}(x,x)-\frac{d-1}{d-2}\right|\le|\mathbf{E}^{\mathbb{T}_d}_\circ[G_{\mathbb{T}_d}(\circ,X_{T_W})]|+|\mathbf{E}^{\mathcal{G}_n}_x[G_{\mathcal{G}_n}(x,X_{T_U})]|+\frac{\mathbf{E}^{\mathcal{G}_n}_x[T_U]}{n}.$$

By (3.10) and (3.11), if K_4 is large enough, then for large enough n:

$$|\mathbf{E}^{\mathbb{T}_d}_\circ[G_{\mathbb{T}_d}(\circ,X_{T_W})]|+|\mathbf{E}^{\mathcal{G}_n}_x[G_{\mathcal{G}_n}(x,X_{T_U})]|\le\log^{-7}n.$$

Note that T_U is stochastically dominated by the hitting time H of $\lfloor K_4\log\log n\rfloor$ by a SRW $(Z_k)_{k\ge0}$ on $\mathbb{Z}$ starting at 0, whose transition probabilities from any vertex are $\frac{d-1}{d}$ towards the right and $1/d$ towards the left. By Markov's exponential inequality, there exists a constant $c>0$ such that for n large enough and every $k>n^{1/10}$,

$$\mathbb{P}(H\ge k)\le\mathbb{P}(Z_k\le\lfloor K_4\log\log n\rfloor)\le\mathbb{P}(Z_k\le(\tfrac{d-2}{d}-1/100)k)\le e^{-ck}.$$

Hence for n large enough, $\mathbf{E}^{\mathcal{G}_n}_x[T_U]\le\mathbb{E}[H]\le n^{1/10}+\sum_{k\ge n^{1/10}}ke^{-ck}\le n^{1/2}$. Thus,

$$\left|G_{\mathcal{G}_n}(x,x)-\frac{d-1}{d-2}\right|\le\log^{-7}n+n^{-1/2}\le\log^{-6}n$$

for large enough n, and this yields (3.8). One proves (3.9) by the same reasoning. □

3.2.2 GFF on $\mathcal{G}_n$

The name "zero-average" for the GFF on $\mathcal{G}_n$ (or on any finite connected graph) comes from the fact that a.s.,

$$\sum_{x\in V_n}\psi_{\mathcal{G}_n}(x)=0 \tag{3.18}$$

since $\mathrm{Var}\left(\sum_{x\in V_n}\psi_{\mathcal{G}_n}(x)\right)=\sum_{x,y\in V_n}G_{\mathcal{G}_n}(x,y)=0$.
This prevents the existence of a domain Markov property. However, there exists a recursive construction of $\psi_{\mathcal{G}_n}$:

Proposition 3.2.3 (Lemma 2.6 in [4]). *Let $A\subsetneq V_n$, $x\in V_n\setminus A$. Write $\sigma(A):=\sigma(\{\psi_{\mathcal{G}_n}(y),\,y\in A\})$. Let $(X_k)_{k\ge0}$ be a SRW on $\mathcal{G}_n$ and let H_A be the hitting time of A. Conditionally on $\sigma(A)$, $\psi_{\mathcal{G}_n}(x)$ is a Gaussian variable, such that*

$$\mathbb{E}^{\mathcal{G}_n}[\psi_{\mathcal{G}_n}(x)|\sigma(A)]=\mathbf{E}^{\mathcal{G}_n}_x[\psi_{\mathcal{G}_n}(X_{H_A})]-\frac{\mathbf{E}^{\mathcal{G}_n}_x[H_A]}{\mathbf{E}^{\mathcal{G}_n}_{\pi_n}[H_A]}\mathbf{E}^{\mathcal{G}_n}_{\pi_n}[\psi_{\mathcal{G}_n}(X_{H_A})] \tag{3.19}$$

and

$$\mathrm{Var}^{\mathcal{G}_n}(\psi_{\mathcal{G}_n}(x)|\sigma(A))=G_{\mathcal{G}_n}(x,x)-\mathbf{E}^{\mathcal{G}_n}_x[G_{\mathcal{G}_n}(x,X_{H_A})]+\frac{\mathbf{E}^{\mathcal{G}_n}_x[H_A]}{\mathbf{E}^{\mathcal{G}_n}_{\pi_n}[H_A]}\mathbf{E}^{\mathcal{G}_n}_{\pi_n}[G_{\mathcal{G}_n}(x,X_{H_A})]. \tag{3.20}$$

Combining this Lemma and the sequential procedure to build $\mathcal{G}_n$, we obtain the following construction.

Proposition 3.2.4 (Joint realization of $\mathcal{G}_n$ and $\psi_{\mathcal{G}_n}$). *A realization of $(\mathcal{G}_n, \psi_{\mathcal{G}_n})$ is given by the following process. Let $(\xi_i)_{i\geq 1}$ be a sequence of i.i.d. standard normal variables. A* ***move*** *consists in:*

- *choosing an unpaired half-edge e and matching it to another unpaired half-edge chosen uniformly at random (independently of $(\xi_i)_{i\geq 1}$), or*
- *choosing $x \in V_n$ and $k \in \mathbb{N}$ such that ξ_k has not yet been attributed, and attributing ξ_k to x.*

At each move, the choice of e, x or k might depend in an arbitrary way on the previous moves, i.e. on the matchings and on the value of the normal variables attributed before, but ***not*** *on the value of the remaining normal variables. Perform moves until all half-edges are paired, and every vertex $x \in V_n$ has received a normal variable, that we denote ξ_x. Erase loops and replace each multiple edge by a single edge.*
To generate $\psi_{\mathcal{G}_n}$, let $x_1, \ldots, x_n$ be the vertices of V_n, listed in the order in which they received their normal variable. Let $\psi_{\mathcal{G}_n}(x_1) := \sqrt{G_{\mathcal{G}_n}(x_1, x_1)}\xi_{x_1}$. For $i = 2, \ldots, n$ successively, define $A_i := \{x_1, \ldots, x_{i-1}\}$. Recall that we write $\sigma(A_i)$ for $\sigma(\{\psi_{\mathcal{G}_n}(y), y \in A_i\})$. Let

$$\psi_{\mathcal{G}_n}(x_i) := \mathbb{E}^{\mathcal{G}_n}[\psi_{\mathcal{G}_n}(x_i)|\sigma(A_i)] + \xi_{x_i}\sqrt{\mathrm{Var}(\psi_{\mathcal{G}_n}(x_i)|\sigma(A_i))}.$$

It might be confusing that $\psi_{\mathcal{G}_n}(x_i)$ appears on both sides of the equation. Note that the conditional expectation and variance on the RHS are $\sigma(A_i)$-measurable random variables.

Proof. Clearly, the graph obtained after pairing all the half-edges is distributed as $\mathcal{G}_n$. ξ_{x_1} is a standard normal variable independent of the realization of $\mathcal{G}_n$. Finally, remark that for every $i \geq 2$, ξ_{x_i} is a standard normal variable, independent of the realization of $\mathcal{G}_n$ and of $\sigma(A_i)$, so that we can conclude by Proposition 3.2.3. $\square$

Last, we prove that the maximum of $|\psi_{\mathcal{G}_n}|$ on $\mathcal{G}_n$ has a subexponential tail.

Lemma 3.2.5 (Tail for the maximum of $|\psi_{\mathcal{G}_n}|$). *Suppose that $\max_{x\in V_n} G_{\mathcal{G}_n}(x,x) \leq K_5$. Then for all $\Delta > 0$, if n is large enough,*

$$\mathbb{P}^{\mathcal{G}_n}\left(\max_{x\in V_n}|\psi_{\mathcal{G}_n}(x)| \geq \log^{2/3} n\right) \leq n^{-\Delta}. \tag{3.21}$$

In particular, by Proposition 3.2.1 and (3.11), w.h.p. $\mathcal{G}_n$ satisfies (3.21).

Proof. Let $N \sim \mathcal{N}(0, K_5)$. If n is large enough, then for all $x \in V_n$,

$$\mathbb{P}^{\mathcal{G}_n}\left(|\psi_{\mathcal{G}_n}(x)| \geq \log^{2/3} n\right) \leq \mathbb{P}^{\mathcal{G}_n}\left(|N| \geq \log^{2/3} n\right) \leq 2\exp\left(-\frac{\log^{4/3} n}{4K_5}\right) \leq n^{-\Delta-1}$$

by Markov's inequality applied to the function $u \mapsto \exp\left(\frac{\log^{2/3} n}{2K_5}u\right)$. By a union bound on all $x \in V_n$, we get $\mathbb{P}^{\mathcal{G}_n}\left(\max_{x\in\mathcal{G}_n}|\psi_{\mathcal{G}_n}(x)| > \log^{2/3} n\right) \leq n^{-\Delta}$. $\square$

3.3 The Gaussian Free Field on $\mathbb{T}_d$

In Section 3.3.1, we characterize $\mathcal{C}_\circ^h$ as a branching process, with a recursive construction (Proposition 3.3.1). Then, in Section 3.3.2, we establish its exponential growth, conditionally on $\{|\mathcal{C}_\circ^h| = +\infty\}$. The main results are Propositions 3.3.4, 3.3.6 and 3.3.8.

3.3.1 $\mathcal{C}_\circ^h$ as a branching process

There is an alternative definition of $\varphi_{\mathbb{T}_d}$, starting from its value at $\circ$ and expanding recursively to its neighbours. It shows that $\mathcal{C}_\circ^h$ is an infinite-type branching process, the type of a vertex x being $\varphi_{\mathbb{T}_d}(x)$.

Proposition 3.3.1 (**Recursive construction of the GFF**,[3]). *Define a Gaussian field φ on $\mathbb{T}_d$ as follows: let $(\xi_y)_{y\in\mathbb{T}_d}$ be a family of i.i.d. $\mathcal{N}(0,1)$ random variables. Let $\varphi(\circ) := \sqrt{\frac{d-1}{d-2}}\xi_\circ$. For every $y \in \mathbb{T}_d \setminus \{\circ\}$, define recursively $\varphi(y) := \sqrt{\frac{d}{d-1}}\xi_y + \frac{1}{d-1}\varphi(\overline{y})$, where $\overline{y}$ is the parent of y. Then*

$$\varphi \overset{d.}{=} \varphi_{\mathbb{T}_d}.$$

Proposition 3.3.1 is the corollary of a more general domain Markov property (see for instance Lemma 1.2 of [123] where it is stated for $\mathbb{Z}^d$, but the proof can readily be adapted to any transient graph). Let $\mathcal{G}$ be a transient graph, $G_\mathcal{G}$ the Green function on it and $\varphi_\mathcal{G}$ the associated GFF. For $U \subsetneq \mathcal{G}$, and $x, y \in \mathcal{G}$, let

$$G_\mathcal{G}^U(x,y) := \mathbf{E}_x^\mathcal{G}\left[\sum_{k=0}^{T_U} \mathbf{1}_{X_k} = y\right]. \tag{3.22}$$

Define the field $\varphi_\mathcal{G}^U$ on $\mathcal{G}$ by $\varphi_\mathcal{G}^U(x) := \varphi_\mathcal{G}(x) - \mathbf{E}_x^\mathcal{G}\left[\varphi_\mathcal{G}(X_{T_U})\right]$ for all $x \in \mathcal{G}$, T_U being the exit time from $\mathcal{G}$.

Proposition 3.3.2 (**Domain Markov property**). *The field $\varphi_\mathcal{G}^U$ is a Gaussian field, independent from $(\varphi_\mathcal{G}(x))_{x\in\mathcal{G}\setminus U}$, with covariances given by $Cov(\phi_\mathcal{G}^U(x), \varphi_\mathcal{G}^U(y)) = G_\mathcal{G}^U(x,y)$.*

We apply it with $\mathcal{G} = \mathbb{T}_d$ and $U = T_y$ the subtree from y in $\mathbb{T}_d$, for every $y \in \mathbb{T}_d \setminus \{\circ\}$, to get Proposition 3.3.1. See [3], (1.4)-(1.9) for details.
Write $\mathbb{P}^{\mathbb{T}_d}$ for the law of $\varphi_{\mathbb{T}_d}$, and $\mathbb{P}_a^{\mathbb{T}_d}$ for $\mathbb{P}^{\mathbb{T}_d}(\,\cdot\,|\varphi_{\mathbb{T}_d}(\circ) = a)$, $a \in \mathbb{R}$ (such conditioning is well-defined, $(\varphi_{\mathbb{T}_d}(x))_{x\in\mathbb{T}_d}$ being a Gaussian process). This construction gives a monotonicity property for $\varphi_{\mathbb{T}_d}$. A set $S \subseteq \mathbb{R}^{\mathbb{T}_d}$ is said to be **increasing** if for any $(\Phi_z^{(1)})_{z\in\mathbb{T}_d}, (\Phi_z^{(2)})_{z\in\mathbb{T}_d} \in \mathbb{R}^{\mathbb{T}_d}$ such that $\Phi_z^{(1)} \leq \Phi_z^{(2)}$ for all $z \in \mathbb{T}_d$, $(\Phi_z^{(1)})_{z\in\mathbb{T}_d} \in S$ only if $(\Phi_z^{(2)})_{z\in\mathbb{T}_d} \in S$. Say that the event $\{\varphi_{\mathbb{T}_d} \in S\}$ is **increasing** if S is increasing.

Lemma 3.3.3 (**Conditional monotonicity**). *If E is an increasing event, then the map $a \mapsto \mathbb{P}_a^{\mathbb{T}_d}(E)$ is non-decreasing on $\mathbb{R}$.*

Proof. Let $a_1, a_2 \in \mathbb{R}$ such that $a_1 > a_2$. It suffices to give a coupling between a GFF $\varphi^{(1)}_{\mathbb{T}_d}$ conditioned on $\varphi^{(1)}_{\mathbb{T}_d}(\circ) = a_1$ and a GFF $\varphi^{(2)}_{\mathbb{T}_d}$ conditioned on $\varphi^{(2)}_{\mathbb{T}_d}(\circ) = a_2$ such that a.s., for every $z \in \mathbb{T}_d$, $\varphi^{(1)}_{\mathbb{T}_d}(z) \geq \varphi^{(2)}_{\mathbb{T}_d}(z)$. To do this, let $(\xi_y)_{y\in\mathbb{T}_d}$ be i.i.d. standard normal variables, and define recursively $\varphi^{(1)}_{\mathbb{T}_d}$ and $\varphi^{(2)}_{\mathbb{T}_d}$ as in Proposition 3.3.1. Then for every $z \in \mathbb{T}_d$ of height $k \geq 0$, $\varphi^{(1)}_{\mathbb{T}_d}(z) = \varphi^{(2)}_{\mathbb{T}_d}(z) + (a_1 - a_2)(d-1)^{-k}$. □

3.3.2 Exponential growth

Let $\mathcal{Z}^h_k := \mathcal{C}^h_\circ \cap \partial B_{\mathbb{T}_d}(\circ, k)$ be the k-th generation of $\mathcal{C}^h_\circ$. We first prove the following:

Proposition 3.3.4. *There exists $\lambda_h > 1$ such that*

$$\lim_{k\to+\infty} \mathbb{P}^{\mathbb{T}_d}(|\mathcal{Z}^h_k| > \lambda^k_h/k^2) = \eta(h)$$

and

$$\lim_{k\to+\infty} \mathbb{P}^{\mathbb{T}_d}(|\mathcal{Z}^h_k| < k\lambda^k_h) = 1.$$

Moreover, $h \mapsto \lambda_h$ is a decreasing homeomorphism from $(-\infty, h_\star)$ to $(1, d-1)$.

We will need the following Lemma (whose proof is immediate from Propositions 3.1 and 3.3 of [132] and Proposition 2.1 (ii) of [3]), from which λ_h originates. Let $\bar{\circ}$ be an arbitrary neighbour of $\circ$. Let $\mathbb{T}^+_d$ be the cone from $\circ$ out of $\bar{\circ}$. Write $\mathcal{C}^{h,+}_\circ := \mathcal{C}^h_\circ \cap \mathbb{T}^+_d$. For $k \geq 1$, let $\mathcal{Z}^{h,+}_k := \mathcal{C}^{h,+}_\circ \cap \partial B_{\mathbb{T}^+_d}(\circ, k)$.

Lemma 3.3.5. *Fix $h < h_\star$. There exist $\lambda_h > 1$ and a function χ_h that is continuous with a positive minimum $\chi_{\min}$ on $[h, +\infty)$, that vanishes on $(-\infty, h)$ and such that*

$$M^h_k := \lambda^{-k}_h \sum_{x\in\mathcal{Z}^{h,+}_k} \chi_h(\varphi_{\mathbb{T}_d}(x))$$

is a martingale w.r.t. the filtration $\mathcal{F}_k := \sigma\left(\varphi_{\mathbb{T}_d}(x), x \in B_{\mathbb{T}^+_d}(\circ, k)\right)$, $k \geq 0$, and has an a.s. limit M^h_∞.

Proof of Proposition 3.3.4. We first establish that $\lim_{k\to+\infty} \mathbb{P}^{\mathbb{T}_d}(|\mathcal{Z}^h_k| > \lambda^k_h/k^2) = \eta(h)$. Clearly,

$$\limsup_{k\to+\infty} \mathbb{P}^{\mathbb{T}_d}(|\mathcal{Z}^h_k| > \lambda^k_h/k^2) \leq \mathbb{P}^{\mathbb{T}_d}(|\mathcal{C}^h_\circ| = +\infty) = \eta(h).$$

Conversely, denote $\mathcal{E}^+ = \{|\mathcal{C}^{h,+}_\circ| = +\infty\}$ and $\mathcal{E}^+_k = \{|\mathcal{Z}^{h,+}_k| \geq \lambda^k_h/k^2\}$. By Theorem 4.3 of [3],

$$\lim_{k\to+\infty} \mathbb{P}^{\mathbb{T}_d}(\mathcal{E}^+_k) = \mathbb{P}^{\mathbb{T}_d}(\mathcal{E}^+) > 0. \tag{3.23}$$

Hence, for any $\varepsilon > 0$, for k large enough,

$$\mathbb{P}^{\mathbb{T}_d}(\mathcal{E}^+_k) \geq \mathbb{P}^{\mathbb{T}_d}(\mathcal{E}^+) - \varepsilon. \tag{3.24}$$

Let $\mathbb{T}_d^-$ be the cone from $\bar{\circ}$ out of $\circ$, $\mathcal{C}_{\bar{\circ}}^{h,-} := \mathcal{C}_{\bar{\circ}}^{h} \cap \mathbb{T}_d^-$, and $\mathcal{Z}_k^{h,-} := \mathcal{C}_{\bar{\circ}}^{h,-} \cap \partial B_{\mathbb{T}_d^-}(\bar{\circ}, k)$ for $k \geq 1$. Let $\mathcal{E}^- = \{|\mathcal{C}_{\bar{\circ}}^{h,-}| = +\infty\}$ and $\mathcal{E}_k^- = \{|\mathcal{Z}_k^{h,-}| \geq \lambda_h^k / k^2\}$. Define $\mathcal{E} := \{\varphi_{\mathbb{T}_d}(\circ) \geq h\} \cap \{\ \mathcal{C}_\circ^{h,+}$ is finite$\}$ and $\mathcal{E}_k := \{\varphi_{\mathbb{T}_d}(\circ) \geq h\} \cap \{\ \mathcal{Z}_k^{h,+} = \emptyset\}$. We have

$$\mathbb{P}^{\mathbb{T}_d}(\mathcal{E}_k^- \cap \mathcal{E}_k) \geq \mathbb{P}^{\mathbb{T}_d}(\mathcal{E}^- \cap \mathcal{E}) - \mathbb{P}^{\mathbb{T}_d}(\mathcal{E}^- \cap \mathcal{E} \cap (\mathcal{E}_k)^c) - \mathbb{P}^{\mathbb{T}_d}(\mathcal{E}^- \cap \mathcal{E} \cap (\mathcal{E}_k^-)^c).$$

Define $M_\infty^{h,-}$ on $\mathcal{C}_{\bar{\circ}}^{h,-}$ as M_∞^h on $\mathcal{C}_\circ^{h,+}$. From the proof of Theorem 4.3 in [3], we get that $\mathbb{P}^{\mathbb{T}_d}(\{M_\infty^{h,-} > 0\} \cap (\mathcal{E}_k^-)^c) \to 0$. And, by Proposition 4.2 in [3], $\mathbb{P}^{\mathbb{T}_d}(\mathcal{E}^- \cap \{M_\infty^{h,-} = 0\}) = 0$, so that $\mathbb{P}^{\mathbb{T}_d}(\mathcal{E}^- \cap (\mathcal{E}_k^-)^c) \to 0$. Moreover, $(\mathcal{E}_k)_{k \geq 0}$ is an increasing sequence of events and $\mathcal{E} = \cup_{k \geq 1} \mathcal{E}_k$, so that $\mathbb{P}^{\mathbb{T}_d}(\mathcal{E} \cap (\mathcal{E}_k)^c) \to 0$. Hence, for k large enough,

$$\mathbb{P}^{\mathbb{T}_d}(\mathcal{E}_k^- \cap \mathcal{E}_k) \geq \mathbb{P}^{\mathbb{T}_d}(\mathcal{E}^- \cap \mathcal{E}) - 2\varepsilon. \tag{3.25}$$

Note that $\{|\mathcal{C}_\circ^h| = +\infty\} = \mathcal{E}^+ \sqcup (\mathcal{E}^- \cap \mathcal{E})$, so that $\mathbb{P}(\mathcal{E}^+) + \mathbb{P}(\mathcal{E}^- \cap \mathcal{E}) = \eta(h)$. And for all $k \geq 2$, $\mathcal{E}_k^+ \sqcup (\mathcal{E}_k^- \cap \mathcal{E}_k) \subseteq \{|\mathcal{Z}_k^h| > \lambda_h^k / k^2\}$, therefore, if k is large enough, by (3.24) and (3.25), one has

$$\mathbb{P}^{\mathbb{T}_d}(|\mathcal{Z}_k^h| > \lambda_h^k / k^2) \geq \eta(h) - 3\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\lim_{k \to +\infty} \mathbb{P}^{\mathbb{T}_d}(|\mathcal{Z}_k^h| > \lambda_h^k / k^2) = \eta(h)$.
Now, we show that $\mathbb{P}^{\mathbb{T}_d}(|\mathcal{Z}_k^h| < k\lambda_h^k) \to 1$. For all $k \geq 1$, by definition of M_k^h,

$$M_k^h \geq \chi_{h,\min} \lambda_h^{-k} \left|\mathcal{Z}_k^h\right|.$$

From the proof of Proposition 3.3 in [132], M_∞^h is a.s. finite. Therefore, $k^{-1} \lambda_h^{-k} \left|\mathcal{Z}_k^h\right| \to 0$ a.s., and

$$\mathbb{P}^{\mathbb{T}_d}(|\mathcal{Z}_k^{h,+}| \geq k\lambda_h^k / 2) \to 0.$$

In the same way, $\mathbb{P}^{\mathbb{T}_d}(|\mathcal{Z}_{k-1}^{h,-}| \geq k\lambda_h^k / 2) \to 0$. Since $\mathcal{Z}_k^h \subseteq \mathcal{Z}_k^{h,+} \cup \mathcal{Z}_{k-1}^{h,-}$, we are done.
The last part of the Proposition comes directly from Propositions 3.1 and 3.3 in [132]. $\square$

Next, we establish finer results on the growth of $\mathcal{C}_\circ^h$. $|\mathcal{C}_\circ^h|$ has exponential moments:

Proposition 3.3.6. *There exists a constant $K_7 > 0$ such that as $k \to +\infty$,*

$$\max_{a \geq h} \mathbb{P}_a^{\mathbb{T}_d}(k \leq |\mathcal{C}_\circ^h| < +\infty) = o(\exp(-K_7 k)). \tag{3.26}$$

Since $\{\mathcal{Z}_k^h \neq \emptyset\} \subseteq \{|\mathcal{C}_\circ^h| \geq k\}$, we have the following straightforward consequence:

Corollary 3.3.7. *For k large enough, for every $a \geq h$,*
$\mathbb{P}_a^{\mathbb{T}_d}(|\mathcal{C}_\circ^h| = +\infty) \leq \mathbb{P}_a^{\mathbb{T}_d}(\mathcal{Z}_k^h \neq \emptyset) \leq \mathbb{P}_a^{\mathbb{T}_d}(|\mathcal{C}_\circ^h| = +\infty) + e^{-K_7 k}$.

In addition, there are large deviation bounds for the growth rate of $\mathcal{Z}_k^h$:

Proposition 3.3.8. *For every $\varepsilon > 0$, there exists $C > 0$ such that for every $k \in \mathbb{N}$ large enough,*

$$\max_{a \geq h} \mathbb{P}_a^{\mathbb{T}_d}(k^{-1} \log |\mathcal{Z}_k^h| \notin [\log(\lambda_h - \varepsilon), \log(\lambda_h + \varepsilon) + k^{-1} \log \chi_h(a)] \mid \mathcal{Z}_k^h \neq \emptyset) \leq \exp(-Ck). \tag{3.27}$$

This also holds when replacing $\mathcal{C}_\circ^h$ by $\mathcal{C}_\circ^{h,+}$, and $\mathcal{Z}_k^h$ by $\mathcal{Z}_k^{h,+}$.

A crucial idea to prove Propositions 3.3.6 and 3.3.8 is to make a finite scaling, in order to get a branching process that is uniformly supercritical w.r.t. to the value of $\varphi_{\mathbb{T}_d}(\circ)$. Indeed, the fact $\lambda_h > 1$ does not ensure that the expected number of children of $\circ$ in $\mathbb{T}_d^+$ (or even in $\mathbb{T}_d$) conditionally on $\varphi_{\mathbb{T}_d}(\circ) = a$ is more than one for every $a \geq h$, in particular if a is small. However, due to the exponential growth of $\mathcal{C}_\circ^h$ (and $\mathcal{C}_\circ^{h,+}$) of Proposition 3.3.4, it turns out that for $\ell \in \mathbb{N}$ large enough, even conditionally on $\varphi_{\mathbb{T}_d}(\circ) = h$, the expected number of vertices in the ℓ-offspring of $\circ$ is more than one, as stated in the Lemma below.

Lemma 3.3.9. *There exists $\ell \in \mathbb{N}$ such that for every $a \geq h$,*

$$\mathbb{E}_a^{\mathbb{T}_d}[|\mathcal{Z}_\ell^h|] \geq \mathbb{E}_a^{\mathbb{T}_d}[|\mathcal{Z}_\ell^{h,+}|] \geq \mathbb{E}_h^{\mathbb{T}_d}[|\mathcal{Z}_\ell^{h,+}|] > 1.$$

We will use it in the proofs of Propositions 3.3.6 and 3.3.8, looking at the branching process whose vertices are those of $\mathcal{C}_\circ^h$ at height 0, ℓ, 2ℓ, etc.

Proof of Lemma 3.3.9. Write $\mathcal{E}_k := \{|\mathcal{Z}_k^{h,+}| \geq \lambda_h^k/k^2\}$. By (3.23), there exists $\varepsilon > 0$ small enough such that for every $k \geq \varepsilon^{-1}$, $\mathbb{P}^{\mathbb{T}_d}(\mathcal{E}_k) \geq \varepsilon$. For a_1 large enough, $\mathbb{P}^{\mathbb{T}_d}(\varphi_{\mathbb{T}_d}(\circ) \geq a_1) < \varepsilon/2$. Note that $\mathcal{E}_k$ is an increasing event, so that by Lemma 3.3.3, the map $a' \mapsto \mathbb{P}_{a'}^{\mathbb{T}_d}(\mathcal{E}_k)$ is non-decreasing. Therefore, for every $a' \geq a_1$ and $k \geq M$, if $\nu \sim \mathcal{N}(0, \frac{d-1}{d-2})$ denotes the law of $\varphi_{\mathbb{T}_d}(\circ)$,

$$\mathbb{P}_{a'}^{\mathbb{T}_d}(\mathcal{E}_k) \geq \mathbb{P}_{a_1}^{\mathbb{T}_d}(\mathcal{E}_k) \geq \int_{-\infty}^{a_1} \mathbb{P}_b^{\mathbb{T}_d}(\mathcal{E}_k)\nu(db) \geq \mathbb{P}^{\mathbb{T}_d}(\mathcal{E}_k) - \mathbb{P}^{\mathbb{T}_d}(\varphi_{\mathbb{T}_d}(\circ) \geq a_1) \geq \varepsilon/2.$$

From the construction of Proposition 3.3.1, it is straightforward that

$$p := \mathbb{P}_h^{\mathbb{T}_d}(\circ \text{ has one child } z \text{ in } \mathcal{C}_\circ^{h,+}, \text{ and } \varphi_{\mathbb{T}_d}(z) \geq a_1) > 0.$$

Hence for $\ell \in \mathbb{N}$ large enough,

$$\mathbb{E}_h^{\mathbb{T}_d}[|\mathcal{Z}_\ell^{h,+}|] \geq \frac{p\varepsilon}{2} \times \frac{\lambda_h^\ell}{\ell^2} > 1.$$

In addition, for every $M \in \mathbb{R}$, $\{|\{\mathcal{Z}_\ell^{h,+}| \geq M\}$ is an increasing event. By Lemma 3.3.3, for every $a \geq h$, $\mathbb{E}_a^{\mathbb{T}_d}[|\mathcal{Z}_\ell^{h,+}|] \geq \mathbb{E}_h^{\mathbb{T}_d}[|\mathcal{Z}_\ell^{h,+}|]$. Since $\mathcal{Z}_\ell^{h,+} \subseteq \mathcal{Z}_\ell^h$ a.s., the conclusion follows. ☐

Proof of Proposition 3.3.6. Fix $a \geq h$, and let $\ell \in \mathbb{N}$ be such that the conclusion of Lemma 3.3.9 holds. Let $F_1 := \partial B_{\mathcal{C}_\circ^h}(\circ, \ell)$. For $j \geq 1$, if $F_j \neq \emptyset$, choose an arbitrary vertex $z_j \in F_j$. Let O_j be the ℓ-offspring of z_j in $\mathcal{C}_\circ^h$ and let $F_{j+1} := O_j \cup F_j \setminus \{z_j\}$. Thus, we explore $\mathcal{C}_\circ^h$ by revealing subtrees of height $\leq \ell$, so that at each step, we see at most $(d-1) + \ldots + (d-1)^\ell \leq d^{\ell+1}$ new vertices. Hence, if $|\mathcal{C}_\circ^h| \geq k$ for some $k \in \mathbb{N}$, there will be at least $\lfloor k/d^{\ell+1} \rfloor$ steps before $\mathcal{C}_\circ^h$ is fully explored.

By Lemma 3.3.3 (applied to $\{|\mathcal{Z}_\ell^{h,+}| \geq k\}$ for every $k \geq 1$), for every $j \geq 1$, $|F_j|$ dominates stochastically a sum S_j of j i.i.d. random variables of law $\rho_{\ell,h} - 1$, where

$$\rho_{\ell,h} \text{ is the law of } |\mathcal{Z}_\ell^{h,+}| \text{ conditionally on } \varphi_{\mathbb{T}_d}(\circ) = h. \tag{3.28}$$

Therefore,

$$\mathbb{P}_a^{\mathbb{T}_d}(k \leq |\mathcal{C}_\circ^h| < +\infty) \leq \sum_{j=\lfloor k/d^{\ell+1}\rfloor}^{+\infty} \mathbb{P}(S_j \leq 0).$$

But a variable of law $\rho_{\ell,h} - 1$ is bounded and has a positive expectation by Lemma 3.3.9, therefore there exist $c, c' > 0$ such that $\mathbb{P}(S_j \leq 0) \leq ce^{-c'j}$ for all $j \geq 1$. Hence,

$$\mathbb{P}_a^{\mathbb{T}_d}(k \leq |\mathcal{C}_\circ^h| < +\infty) \leq c \sum_{j=\lfloor k/d^{\ell+1}\rfloor}^{+\infty} e^{-c'j} \leq \frac{c\exp(-c'\lfloor k/d^{\ell+1}\rfloor)}{1-e^{-c'}}$$

and (3.26) follows. □

Proof of Proposition 3.3.8. Let $\varepsilon > 0$. By definition of M_k^h and Lemma 3.3.5,

$$\{|\mathcal{Z}_k^{h,+}| \geq \chi_h(a)(\lambda_h+\varepsilon)^k\} \subseteq \{M_k^h \geq \lambda_h^{-k}\chi_{h,\min}\chi_h(a)(\lambda_h+\varepsilon)^k\}$$

so that by Markov's inequality, for any $a \geq h$,

$$\begin{aligned}\mathbb{P}_a^{\mathbb{T}_d}(|\mathcal{Z}_k^{h,+}| \geq \chi_h(a)(\lambda_h+\varepsilon)^k) &\leq \mathbb{P}_a^{\mathbb{T}_d}\left(M_k^h \geq \chi_{h,\min}\chi_h(a)\left(\frac{\lambda_h+\varepsilon}{\lambda_h}\right)^k\right)\\ &\leq \chi_{h,\min}^{-1}\left(\frac{\lambda_h}{\lambda_h+\varepsilon}\right)^k \chi_h(a)^{-1}\mathbb{E}_a^{\mathbb{T}_d}[M_k^h]\\ &\leq \chi_{h,\min}^{-1}\left(\frac{\lambda_h}{\lambda_h+\varepsilon}\right)^k.\end{aligned}$$

This yields the upper large deviation for $\mathcal{Z}_k^{h,+}$ (and for $\mathcal{Z}_k^h$, using the facts that $\mathcal{Z}_k^h \subseteq \mathcal{Z}_k^{h,+} \cup \mathcal{Z}_{k-1}^{h,-}$ and that $|\mathcal{Z}_{k-1}^{h,-}|$ and $|\mathcal{Z}_{k-1}^{h,+}|$ have the same law).

It remains to prove that for some $C > 0$ and k large enough,

$$\max_{a\geq h} \mathbb{P}_a^{\mathbb{T}_d}(k^{-1}\log|\mathcal{Z}_k^h| \leq \log(\lambda_h - \varepsilon) \,|\, \mathcal{Z}_k^h \neq \emptyset) \leq \exp(-Ck). \tag{3.29}$$

We proceed in two steps. We first initiate the growth of $\mathcal{C}_\circ^h$ by showing that if $\mathcal{Z}_{\ell n} \neq \emptyset$, the probability that $|\mathcal{Z}_{\ell n}| = o(n)$ decays exponentially with n, where ℓ is such that Lemma 3.3.9 holds. Then, if $\mathcal{Z}_{\ell n}$ has at least $\Theta(n)$ vertices, each of them has a positive probability to have a Kn-offspring of size at least $\lambda_h^{Kn}/(Kn)^3 \geq (\lambda_h - \varepsilon)^k$ with $k := (K+\ell)n$ and for a large enough constant K, independently of the others vertices. Hence the probability that $|\mathcal{Z}_k| \leq (\lambda_h - \varepsilon)^k$ decays exponentially with n, and thus with k.

First step. Recall the exploration of $\mathcal{C}_\circ^h$ of the proof of Proposition 3.3.6, but perform it in a breadth-first way: reveal first the ℓ-offspring of $\circ$, then the ℓ-offspring of each vertex of $\mathcal{Z}_\ell^h$, then the ℓ-offspring of each vertex of $\mathcal{Z}_{2\ell}^h$, and so on. For $n \geq 1$, if $\mathcal{Z}_{\ell n}^h \neq \emptyset$, let $j+1$ be the first step at which we explore the offspring of a vertex of $\mathcal{Z}_{\ell n}$. Note that $j \geq n$. As in the proof of Proposition 3.3.6, there exist $\epsilon, c, c' > 0$ such that $\mathbb{P}(S_i \leq \epsilon i) \leq ce^{-c'i}$ for every $i \geq 1$. Hence, for every $n \geq 1$,

$$\min_{a\geq h} \mathbb{P}_a^{\mathbb{T}_d}(|\mathcal{Z}_{\ell n}^h| \geq \epsilon n \,|\, \mathcal{Z}_{\ell n}^h \neq \emptyset) \geq 1 - \sum_{i\geq n} ce^{-c'i} \geq 1 - \frac{c}{1-e^{-c'}}e^{-c'n}. \tag{3.30}$$

Second step. Let K be a positive integer constant, and let F be the set of vertices $z \in \mathcal{Z}^h_{\ell n}$ such that the Kn-offspring of z has at least λ_h^{Kn}/n^3 vertices. This step mainly comes down to showing that the probability that F is empty decays exponentially with n.
Define the events

$$\mathcal{E}_n := \{|\mathcal{Z}^{h,+}_{Kn}| \geq \lambda_h^{Kn}/n^3\} \text{ and } \mathcal{E}'_n := \{|\mathcal{Z}^{h,+}_{Kn-1}| \geq \lambda_h^{Kn}/n^3\}.$$

We first show that

$$p := \min_{a \geq h} \mathbb{P}^{\mathbb{T}_d}_a(\mathcal{E}_n) > 0. \tag{3.31}$$

By (3.23), $\liminf_{n\to+\infty} \mathbb{P}^{\mathbb{T}_d}(\mathcal{E}'_n) =: p' > 0$. Let a_1 be such that $\mathbb{P}(\varphi_{\mathbb{T}_d}(\circ) \geq a_1) < p'/4$. For n large enough,

$$\textstyle\int_{a\geq h} \mathbb{P}^{\mathbb{T}_d}_a(\mathcal{E}'_n)\nu(da) > p'/2, \text{ hence } \int_h^{a_1} \mathbb{P}^{\mathbb{T}_d}_a(\mathcal{E}'_n)\nu(da) > p'/4,$$

where we recall that ν is the density of $\varphi_{\mathbb{T}_d}(\circ)$. Since $\mathcal{E}'_n$ is an increasing event, by Lemma 3.3.3:

$$\min_{a\geq a_1} \mathbb{P}^{\mathbb{T}_d}_a(\mathcal{E}'_n) \geq p'/4.$$

By Lemma 3.3.3 again,

$$\min_{a\geq h} \mathbb{P}^{\mathbb{T}_d}_a(\exists z \in \mathcal{Z}^{h,+}_1,\, \varphi_{\mathbb{T}_d}(z) \geq a_1) = \mathbb{P}^{\mathbb{T}_d}_h(\exists z \in \mathcal{Z}^{h,+}_1,\, \varphi_{\mathbb{T}_d}(z) \geq a_1) =: p'' > 0.$$

Hence $p \geq p'p''/4$ and (3.31) is proved.
Note that $|F| \underset{st.}{\geq} \mathrm{Bin}(|\mathcal{Z}^h_{\ell n}|, p'')$. Thus by (3.30), for n large enough,

$$\min_{a\geq h} \mathbb{P}^{\mathbb{T}_d}_a(|F| \geq 1 | \mathcal{Z}^h_{\ell n} \neq \emptyset) \geq 1 - \mathbb{P}^{\mathbb{T}_d}_a(|\mathcal{Z}^h_{\ell n}| \leq \epsilon n | \mathcal{Z}^h_{\ell n} \neq \emptyset) - (1-p'')^{\epsilon j} \geq 1 - ce^{-c'n},$$

up to changing the values of the constants c and c'. Therefore,

$$\max_{a\geq h} \mathbb{P}^{\mathbb{T}_d}_a\left(|\mathcal{Z}^h_{(K+\ell)n}| < \lambda_h^{Kn}/n^3 \,|\, \mathcal{Z}^h_{\ell n} \neq \emptyset\right) \leq ce^{-c'j} \leq ce^{-c'n}.$$

If K is large enough, then for n large enough, $\lambda_h^{Kn}/n^3 > (\lambda_h - \varepsilon)^{(K+\ell)(n+1)}$, so that

$$\max_{a\geq h} \mathbb{P}^{\mathbb{T}_d}_a\left(|\mathcal{Z}^h_{(K+\ell)n}| < (\lambda_h - \varepsilon)^{(K+\ell)(n+1)} \,|\, \mathcal{Z}^h_{\ell n} \neq \emptyset\right) \leq ce^{-c'n}.$$

We adjust the conditionning: $\{\mathcal{Z}^h_{(K+\ell)n} \neq \emptyset\} \subseteq \{\mathcal{Z}^h_{\ell n} \neq \emptyset\}$, and $|\mathcal{Z}^h_{(K+\ell)n}| < (\lambda_h - \varepsilon)^{(K+\ell)(n+1)}$ on $\{\mathcal{Z}^h_{\ell n} \neq \emptyset\} \setminus \{\mathcal{Z}^h_{(K+\ell)n} \neq \emptyset\}$. Therefore, there exists $n_0 \geq 1$ such that for all $n \geq n_0$,

$$\max_{a\geq h} \mathbb{P}^{\mathbb{T}_d}_a\left(|\mathcal{Z}^h_{(K+\ell)n}| < (\lambda_h - \varepsilon)^{(K+\ell)(n+1)} \,|\, \mathcal{Z}^h_{(K+\ell)n} \neq \emptyset\right) \leq c\exp(-c'(K+\ell)(n+1)), \tag{3.32}$$

where the new value of c' depends on the constants K and ℓ. This yields (3.29) for large enough multiples of $K+\ell$. One can readily replace each $\mathcal{Z}^h_m$ by $\mathcal{Z}^{h,+}_m$ in this reasoning (for any $m \geq 1$), to get the same result for $\mathcal{Z}^{h,+}_{(K+\ell)n}$ instead of $\mathcal{Z}^h_{(K+\ell)n}$.
It remains to show the result for non multiples of $K+\ell$. Let $k \geq (K+\ell)n_0$. Write $k =$

$(K+\ell)n+m$, with $0 \le m \le (K+\ell)-1$. Note that on $\{\mathcal{Z}_k^h \neq \emptyset\}$, $\mathcal{Z}_m^h$ has at least one vertex whose $(k-m)$-offspring is not empty. Hence

$$\begin{aligned}\max_{a\geq h} \mathbb{P}_a^{\mathbb{T}_d}\left(|\mathcal{Z}_k^h| < (\lambda_h-\varepsilon)^k \mid \mathcal{Z}_k^h \neq \emptyset\right) &\le \max_{a\geq h} \mathbb{P}_a^{\mathbb{T}_d}\left(|\mathcal{Z}_{k-m}^{h,+}| < (\lambda_h-\varepsilon)^k \mid \mathcal{Z}_{k-m}^{h,+} \neq \emptyset\right)\\ &\le \max_{a\geq h} \mathbb{P}_a^{\mathbb{T}_d}\left(|\mathcal{Z}_{k-m}^{h,+}| < (\lambda_h-\varepsilon)^{(k-m)+(K+\ell)} \mid \mathcal{Z}_{k-m}^{h,+} \neq \emptyset\right)\\ &\le c\exp(-c'(K+\ell)(n+1))\\ &\le ce^{-c'k},\end{aligned}$$

where the third inequality comes from (3.32). Adapting this last computation for $\mathcal{Z}_k^{h,+}$ is immediate. This concludes the proof of (3.27). □

3.4 Approximate recursive construction of $\psi_{\mathcal{G}_n}$

Let $\kappa > 0$ be a constant, and let

$$a_n := \lfloor \kappa \log_{d-1} \log n \rfloor. \tag{3.33}$$

The following statement is the main result of this section. It shows that a recursive construction of $\psi_{\mathcal{G}_n}$, under some assumptions on the subset $A \subseteq V_n$ of vertices where $\psi_{\mathcal{G}_n}$ is already known, is very close to the construction of $\varphi_{\mathbb{T}_d}$ in Proposition 3.3.1. It is a crucial tool for comparing $\psi_{\mathcal{G}_n}$ and $\varphi_{\mathbb{T}_d}$ in the exploration in the next section. It is analogous to Proposition 2.7 of [4], where the assumptions on A are slightly different: they are suited to a deterministic d-regular graph satisfying (I) and (II), while ours will be adapted to an annealed exploration, where the randomness of $\mathcal{G}_n$ plays a role. Our proof is very similar, but we feel that the general argument is sufficiently subtle and interesting to merit a full account.

Proposition 3.4.1. *If the constant κ from (3.33) is large enough, then the following holds for n large enough. Assume that $\mathcal{G}_n$ is a good graph as defined in Proposition 3.2.1, and that $A \subseteq V_n$ satisfies*

- $|A| \le n \log^{-8} n$,
- $\mathrm{tx}(B_{\mathcal{G}_n}(A, a_n)) = \mathrm{tx}(A)$, *and*
- $\max_{z\in A} |\psi_{\mathcal{G}_n}(z)| \le \log^{2/3} n$.

For every $y \in \partial B_{\mathcal{G}_n}(A,1)$, writing $\overline{y}$ for the unique neighbour of y in A, we have:

$$\left|\mathbb{E}^{\mathcal{G}_n}[\psi_{\mathcal{G}_n}(y)|\sigma(A)] - \frac{1}{d-1}\psi_{\mathcal{G}_n}(\overline{y})\right| \le \log^{-3} n \tag{3.34}$$

and

$$\left|\mathrm{Var}^{\mathcal{G}_n}(\psi_{\mathcal{G}_n}(y)|\sigma(A)) - \frac{d}{d-1}\right| \le \log^{-4} n. \tag{3.35}$$

We stress that the result holds deterministically in $\mathcal{G}_n$.

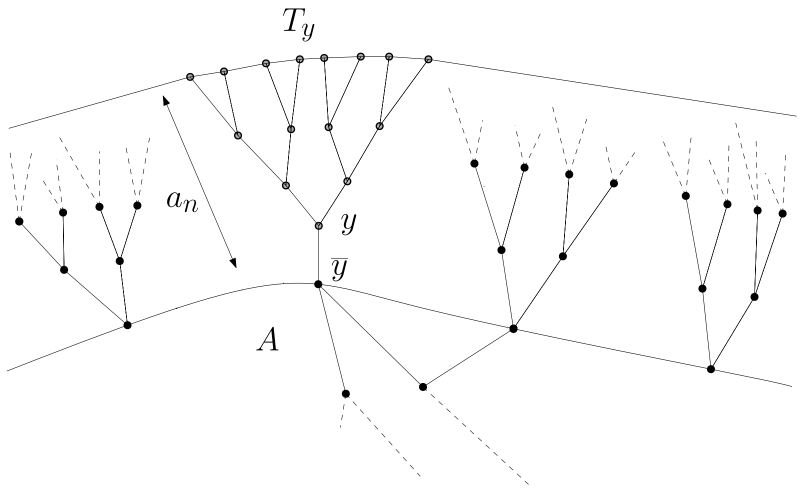

Figure 1. The unicity of $\overline{y}$ comes from the fact that when building $B_{\mathcal{G}_n}(A, a_n)$ from A, no cycle appears and no connected components of A join, since $\mathrm{tx}(B_{\mathcal{G}_n}(A, a_n)) = \mathrm{tx}(A)$.

In the proof, we will need the fact below, which is a consequence of Lemma 3.3 in [140] and of the following observation. If $(X_j)_{j\geq 0}$ is a SRW on $\mathbb{T}_d$, then the trajectory of its height $(\mathfrak{h}_{\mathbb{T}_d}(X_j))_{j\geq 0}$ is distributed as a random walk on $\mathbb{N}\cup\{0\}$ with transition probabilities $1/d$ towards the left neighbour and $(d-1)/d$ towards the right neighbour, and reflected at 0.

Lemma 3.4.2 (Geometric repulsion). *Let $s \in \mathbb{N}$, and $A \subseteq V_n$ such that $\mathrm{tx}(B_{\mathcal{G}_n}(A, s)) = \mathrm{tx}(A)$. Let $x \in V_n \setminus B_{\mathcal{G}_n}(A, s)$, let $(X_j)_{j\geq 0}$ be a SRW started at x and τ its first hitting time of A. Then τ dominates stochastically a geometric random variable of parameter $(d-1)^{-s}$.*

Proof of Proposition 3.4.1. Let us first prove (3.34). By Proposition 3.2.3,

$$\mathbb{E}^{\mathcal{G}_n}[\psi_{\mathcal{G}_n}(y)|\sigma(A)] = \mathbf{E}_y^{\mathcal{G}_n}\left[\psi_{\mathcal{G}_n}(X_{H_A})\right] - \frac{\mathbf{E}_y^{\mathcal{G}_n}[H_A]}{\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[H_A]}\mathbf{E}_{\pi_n}^{\mathcal{G}_n}\left[\psi_{\mathcal{G}_n}(X_{H_A})\right],$$

where $(X_j)_{j\geq 0}$ is a SRW on $\mathcal{G}_n$. Write T_y for $B_{\mathcal{G}_n}(y, \overline{y}, a_n - 1)$, which is a tree rooted at y by our assumptions on A and y (it consists of the paths of length $a_n - 1$ starting at y and not going through $\overline{y}$).
Let ∂T_y be the $(a_n - 1)$-offspring of y in T_y. Let τ be the hitting time of ∂T_y by (X_j). Then splitting the first term of the RHS above into two terms, we obtain

$$\begin{aligned}\mathbb{E}^{\mathcal{G}_n}[\psi_{\mathcal{G}_n}(y)|\sigma(A)] = \mathbf{E}_y^{\mathcal{G}_n}\left[\psi_{\mathcal{G}_n}(X_{H_A})\mathbf{1}_{H_A\leq\tau}\right] + \mathbf{E}_y^{\mathcal{G}_n}\left[\psi_{\mathcal{G}_n}(X_{H_A})\mathbf{1}_{H_A>\tau}\right] \\ - \frac{\mathbf{E}_y^{\mathcal{G}_n}[H_A]}{\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[H_A]}\mathbf{E}_{\pi_n}\left[\psi_{\mathcal{G}_n}(X_{H_A})\right].\end{aligned} \tag{3.36}$$

On $\{H_A \leq \tau\}$, $X_{H_A} = \overline{y}$. And, $\mathbf{P}_y^{\mathcal{G}_n}(H_A \leq \tau)$ is the probability that a SRW on $\mathbb{Z}$ started at 1 with transition probabilities $1/d$ towards the left and $(d-1)/d$ towards the right hits 0 before

a_n. It is classical that its difference with $1/(d-1)$ decays exponentially with a_n (see for instance Theorem 4.8.9 in [72]). Hence, if κ is large enough,

$$\left|\mathbf{P}_y^{\mathcal{G}_n}(H_A \leq \tau) - \frac{1}{d-1}\right| \leq \log^{-6} n, \tag{3.37}$$

and since $\max_{z\in A} |\psi_{\mathcal{G}_n}(z)| \leq \log^{2/3} n$,

$$\left|\mathbf{E}_y^{\mathcal{G}_n}\left[\psi_{\mathcal{G}_n}\left(X_{H_A}\right)\mathbf{1}_{H_A\leq\tau}\right] - \frac{1}{d-1}\psi_{\mathcal{G}_n}(\overline{y})\right| \leq \log^{-4} n.$$

Therefore, to establish (3.34), it is enough to show that

$$\left|\mathbf{E}_y^{\mathcal{G}_n}\left[\psi_{\mathcal{G}_n}\left(X_{H_A}\right)\mathbf{1}_{H_A>\tau}\right] - \frac{d-2}{d-1}\mathbf{E}_{\pi_n}^{\mathcal{G}_n}\left[\psi_{\mathcal{G}_n}\left(X_{H_A}\right)\right]\right| \leq 3\log^{-5} n \tag{3.38}$$

and

$$\left|\frac{d-2}{d-1} - \frac{\mathbf{E}_y^{\mathcal{G}_n}[H_A]}{\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[H_A]}\right|\mathbf{E}_{\pi_n}^{\mathcal{G}_n}\left[\psi_{\mathcal{G}_n}\left(X_{H_A}\right)\right] \leq \log^{-5} n. \tag{3.39}$$

By the strong Markov property, letting $p_z := \mathbf{P}_y^{\mathcal{G}_n}(H_A > \tau, X_{H_{\partial T_y}} = z)$ for $z \in \partial T_y$, we have

$$\mathbf{E}_y^{\mathcal{G}_n}\left[\psi_{\mathcal{G}_n}\left(X_{H_A}\right)\mathbf{1}_{H_A>\tau}\right] = \textstyle\sum_{z\in\partial T_y} p_z \mathbf{E}_z^{\mathcal{G}_n}\left[\psi_{\mathcal{G}_n}\left(X_{H_A}\right)\right].$$

For all $z \in \partial T_y$, by Lemma 3.4.2 (with $s = a_n$), if κ is large enough, then for large enough n,

$$\mathbf{P}_z^{\mathcal{G}_n}(H_A < \log^2 n) \leq 1 - (1-(d-1)^{-a_n})^{\log^2 n} \leq 2\log^2 n\ (d-1)^{-a_n} \leq \log^{-6} n.$$

Therefore,

$$\begin{aligned}\left|\mathbf{E}_y^{\mathcal{G}_n}\left[\psi_{\mathcal{G}_n}\left(X_{H_A}\right)\mathbf{1}_{H_A>\tau}\right] - \sum_{z\in\partial T_y} p_z \sum_{z'\in V_n} \mathbf{P}_z^{\mathcal{G}_n}(X_{\lfloor\log^2 n\rfloor} = z')\mathbf{E}_{z'}^{\mathcal{G}_n}\left[\psi_{\mathcal{G}_n}\left(X_{H_A}\right)\right]\right| \\ \leq \mathbf{P}_z^{\mathcal{G}_n}(H_A < \log^2 n)\max_{z\in A}|\psi_{\mathcal{G}_n}(z)| \\ \leq \log^{-5} n.\end{aligned}$$

By Corollary 2.1.5 of [126] and (I), $\left|\mathbf{P}_z^{\mathcal{G}_n}\left(X_{\lfloor\log^2 n\rfloor} = z'\right) - \pi_n(z')\right| \leq e^{-\lambda_{\mathcal{G}_n}\log^2 n} \leq \frac{1}{n^2}$ for n large enough and all $z, z' \in V_n$, so that

$$\left|\mathbf{E}_y^{\mathcal{G}_n}\left[\psi_{\mathcal{G}_n}\left(X_{H_A}\right)\mathbf{1}_{H_A>\tau}\right] - \sum_{z\in\partial T_y} p_z \mathbf{E}_{\pi_n}^{\mathcal{G}_n}\left[\psi_{\mathcal{G}_n}\left(X_{H_A}\right)\right]\right| \leq 2\log^{-5} n.$$

Finally, note that $\left|\sum_{z\in\partial T_y} p_z - \frac{d-2}{d-1}\right| = \left|\mathbf{P}_y^{\mathcal{G}_n}(H_A > \tau) - \frac{d-2}{d-1}\right| \leq \log^{-6} n$ by (3.37), and this yields (3.38).

Let τ' be the hitting time of $\partial T_y \cup \{\overline{y}\}$ by (X_j). By the strong Markov property again, we have

$$\mathbf{E}_y^{\mathcal{G}_n}[H_A] = \mathbf{E}_y^{\mathcal{G}_n}\left[\tau'\right] + \sum_{z\in\partial T_y} \mathbf{P}_y^{\mathcal{G}_n}(X_{\tau'} = z)\mathbf{E}_z^{\mathcal{G}_n}[H_A]. \tag{3.40}$$

If $(Z_j)_{j\geq 0}$ is a SRW on $\mathbb{Z}$ starting at 1 with transition probabilities $1/d$ towards the left and $(d-1)/d$ towards the right, then τ' has the law of the hitting time of $\{0, a_n\}$ by (Z_j). Hence, for all $k \geq 0$, $\mathbb{P}_y(\tau' \geq k) \leq \mathbb{P}(Z_k \leq a_n)$. But Z_j is the sum of j i.i.d. increments that are bounded a.s. and with mean $(d-2)/d > 0$. We apply the exponential Markov inequality and obtain that for some $\gamma > 0$ and n large enough, for all $k \geq \log n$, $\mathbb{P}(Z_k \leq a_n) \leq \exp(-\gamma k)$. Thus,

$$\mathbf{E}_y^{\mathcal{G}_n}\left[\tau'\right] \leq \log n \;\; \mathbf{P}_y^{\mathcal{G}_n}(\tau' \leq \log n) + \sum_{k\geq \log n} k\exp(-\gamma k) \leq 2\log n. \tag{3.41}$$

By (3.20) of [140], $\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[H_A] \geq \frac{n}{4|A|} \geq \log^8 n/4$, so that dividing by $\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[H_A]$ in (3.40), we get

$$\left|\frac{\mathbf{E}_y^{\mathcal{G}_n}[H_A]}{\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[H_A]} - \sum_{z\in\partial T_y} p'_z \frac{\mathbf{E}_z^{\mathcal{G}_n}[H_A]}{\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[H_A]}\right| \leq \frac{8\log n}{\log^8 n} \leq \log^{-6} n,$$

where $p'_z := \mathbf{P}_y^{\mathcal{G}_n}(X_{\tau'} = z)$. By (3.37), $\left|\sum_{z\in\partial T_y} p'_z - \frac{d-2}{d-1}\right| \leq \log^{-6} n$. Therefore,

$$\left|\frac{\mathbf{E}_y^{\mathcal{G}_n}[H_A]}{\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[H_A]} - \frac{d-2}{d-1}\right| \leq 2\log^{-6} n + \max_{z\in\partial T_y}\left|\frac{\mathbf{E}_z^{\mathcal{G}_n}[H_A]}{\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[H_A]} - 1\right|.$$

To conclude the proof of (3.39), and thus of (3.34), it is enough to show that

$$\max_{z\in\partial T_y}\left|\frac{\mathbf{E}_z^{\mathcal{G}_n}[H_A]}{\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[H_A]} - 1\right| \leq 5\log^{-6} n. \tag{3.42}$$

We adapt for this the proof of Proposition 3.5 in [140]. For all $z \in \partial T_y$, we can write $\mathbf{E}_z^{\mathcal{G}_n}[H_A] \leq \mathbf{E}_z^{\mathcal{G}_n}\left[H_A \mathbf{1}_{H_A < \log^2 n}\right] + \sum_{z'\in V_n} \mathbf{P}_z^{\mathcal{G}_n}(X_{\lfloor \log^2 n\rfloor} = z', H_A \geq \log^2 n)(\mathbf{E}_{z'}^{\mathcal{G}_n}[H_A] + \log^2 n)$, hence using Corollary 2.1.5 of [126] and the fact that π_n is uniform on V_n,

$$\mathbf{E}_z^{\mathcal{G}_n}[H_A] \leq \log^2 n + \sum_{z'\in V_n}\Big(\pi_n(z') + e^{-\lambda_{\mathcal{G}_n}\log^2 n}\Big)\,\mathbf{E}_{z'}^{\mathcal{G}_n}[H_A] \leq \log^2 n + (1 + ne^{-\lambda_{\mathcal{G}_n}\log^2 n})\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[H_A]. \tag{3.43}$$

Recall that $\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[H_A] \geq \log^8 n/4$. For n large enough, we have by (3.43):

$$\frac{\mathbf{E}_z^{\mathcal{G}_n}[H_A]}{\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[H_A]} \leq \frac{4\log^2 n}{\log^8 n} + 1 + ne^{-\lambda_{\mathcal{G}_n}\log^2 n} \leq 1 + 5\log^{-6} n. \tag{3.44}$$

Conversely,

$$\begin{aligned}
\mathbf{E}_z^{\mathcal{G}_n}[H_A] \geq & \sum_{z'\in V_n} \mathbf{P}_z^{\mathcal{G}_n}(X_{\lfloor \log^2 n\rfloor} = z')\mathbf{E}_{z'}^{\mathcal{G}_n}[H_A] - \sum_{z'\in V_n} \mathbf{P}_z^{\mathcal{G}_n}(X_{\lfloor \log^2 n\rfloor} = z', H_A \leq \log^2 n)\mathbf{E}_{z'}^{\mathcal{G}_n}[H_A] \\
\geq & \sum_{z'\in V_n}(\pi_n(z') - e^{-\lambda_{\mathcal{G}_n}\log^2 n})\mathbf{E}_{z'}^{\mathcal{G}_n}[H_A] - \mathbf{P}_z^{\mathcal{G}_n}(H_A \leq \log^2 n)\sup_{z'\in V_n}\mathbf{E}_{z'}^{\mathcal{G}_n}[H_A] \\
\geq & (1 - ne^{-\lambda_{\mathcal{G}_n}\log^2 n})\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[H_A] - \mathbf{P}_z^{\mathcal{G}_n}(H_A \leq \log^2 n)\sup_{z'\in V_n}\mathbf{E}_{z'}^{\mathcal{G}_n}[H_A].
\end{aligned}$$

(3.43) and (3.44) are in fact true for all $z \in V_n$, so that for large enough n,

$$\begin{aligned}
\frac{\mathbf{E}_z^{\mathcal{G}_n}[H_A]}{\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[H_A]} \geq & 1 - ne^{-\lambda_{\mathcal{G}_n}\log^2 n} - \mathbf{P}_z^{\mathcal{G}_n}(H_A \leq \log^2 n)(1 + 5\log^{-6} n) \\
\geq & 1 - 2\log^{-6} n,
\end{aligned}$$

provided that κ is large enough: by Lemma 3.4.2 applied as below (3.39), we have

$$\mathbf{P}_z^{\mathcal{G}_n}(H_A \le \log^2 n) \le \log^{-6} n \text{ for every } z \in \partial T_y.$$

Here lies the main difference with Proposition 3.5 of [140], where there might be more cycles in $B_{\mathcal{G}_n}(A, a_n)$ than in A (i.e. the trees planted on the boundary of A might intersect). One has to use a weaker version of Lemma 3.4.2, and the lower bound becomes $1 - |A|\log^{-\Delta} n$, Δ being any positive constant (κ has to be large enough w.r.t. Δ).

Thus, the proof of (3.34) is complete (note that the required lower bounds on κ given by Lemma 3.4.2 are uniform in y and A).

We prove (3.35) in the same fashion. Note in particular that by (3.20),

$$\mathrm{Var}^{\mathcal{G}_n}(\psi_{\mathcal{G}_n}(y)|\sigma(A)) = G_{\mathcal{G}_n}(y,y) - \mathbf{E}_y^{\mathcal{G}_n}\left[G_{\mathcal{G}_n}\left(y, X_{H_A}\right)\right] + \frac{\mathbf{E}_y^{\mathcal{G}_n}[H_A]}{\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[H_A]}\mathbf{E}_{\pi_n}^{\mathcal{G}_n}\left[G_{\mathcal{G}_n}\left(y, X_{H_A}\right)\right]$$

so that we get, noticing that $H_A \ge \tau'$ (the hitting time of $\partial T_y \cup \{\overline{y}\}$) a.s.,

$$\begin{aligned}\left|\mathrm{Var}^{\mathcal{G}_n}(\psi_{\mathcal{G}_n}(y)|\sigma(A)) - \frac{d}{d-1}\right| &\le \left|G_{\mathcal{G}_n}(y,y) - \mathbf{E}_y^{\mathcal{G}_n}\left[G_{\mathcal{G}_n}\left(y, X_{H_A}\right)\mathbf{1}_{H_A=\tau'}\right] - \frac{d}{d-1}\right| \\ &+ \left|\mathbf{E}_y^{\mathcal{G}_n}\left[G_{\mathcal{G}_n}\left(y, X_{H_A}\right)\mathbf{1}_{H_A>\tau'}\right] - \frac{\mathbf{E}_y^{\mathcal{G}_n}[H_A]}{\mathbf{E}_{\pi_n}[H_A]}\mathbf{E}_{\pi_n}^{\mathcal{G}_n}\left[G_{\mathcal{G}_n}\left(y, X_{H_A}\right)\right]\right|\end{aligned} \tag{3.45}$$

We have

$$\begin{aligned}\mathbf{E}_y^{\mathcal{G}_n}\left[G_{\mathcal{G}_n}\left(y, X_{H_A}\right)\mathbf{1}_{H_A=\tau'}\right] &= \mathbf{E}_y^{\mathcal{G}_n}\left[G_{\mathcal{G}_n}\left(y, X_{\tau'}\right)\mathbf{1}_{H_A=\tau'}\right] \\ &= \mathbf{E}_y^{\mathcal{G}_n}\left[G_{\mathcal{G}_n}\left(y, X_{\tau'}\right)\right] - \mathbf{E}_y^{\mathcal{G}_n}\left[G_{\mathcal{G}_n}\left(y, X_{\tau'}\right)\mathbf{1}_{H_A>\tau'}\right].\end{aligned}$$

Remark that

$$\left|\mathbf{E}_y^{\mathcal{G}_n}\left[G_{\mathcal{G}_n}\left(y, X_{\tau'}\right)\mathbf{1}_{H_A>\tau'}\right]\right| \le \max_{z\in\partial T_y}|G_{\mathcal{G}_n}(y,z)| \le \log^{-5} n$$

if κ is large enough, by (3.11). Now, by (3.17) applied to the tree T_y (note that $T_{T_y} = \tau'$), we get

$$|G_{\mathcal{G}_n}(y,y) - \mathbf{E}_y^{\mathcal{G}_n}\left[G_{\mathcal{G}_n}\left(y, X_{\tau'}\right)\right] - G_{\mathcal{G}_n}^{T_y}(y,y)| \le \frac{\mathbf{E}_y^{\mathcal{G}_n}[\tau']}{n}$$

By (3.41), $\frac{\mathbf{E}_y^{\mathcal{G}_n}[\tau']}{n} = O(\log^{-5} n)$, therefore, the first term of the RHS of (3.45) is

$$|G_{\mathcal{G}_n}^{T_y}(y,y) - \tfrac{d}{d-1}| + O(\log^{-5} n).$$

But T_y is isomorphic to $B := B_{\mathbb{T}_d^+}(\circ, a_n - 1)$, so that

$$G_{\mathcal{G}_n}^{T_y}(y,y) = G_{\mathbb{T}_d}^B(\circ,\circ) = G_{\mathbb{T}_d}(\circ,\circ) - \mathbf{P}_\circ^{\mathbb{T}_d}(T_B = \overline{\circ})G_{\mathbb{T}_d}(\circ,\overline{\circ}) - \mathbf{P}_\circ^{\mathbb{T}_d}(T_B \ne \overline{\circ})G_{\mathbb{T}_d}(\circ, z)$$

for any $z \in \mathbb{T}_d^+$ at distance $a_n - 1$ of $\circ$, by cylindrical symmetry of $\mathbb{T}_d^+$. By (3.10), if κ is large enough:

$$G_{\mathcal{G}_n}^{T_y}(y,y) = \tfrac{d-1}{d-2} - \tfrac{\mathbf{P}_\circ^{\mathbb{T}_d}(T_B=\overline{\circ})}{d-2} + O(\log^{-5} n).$$

By the same reasoning as that leading to (3.37), $\mathbf{P}_\circ^{\mathbb{T}_d}(T_B = \bar{\circ}) = \frac{1}{d-1} + O(\log^{-5} n)$ for κ large enough, hence

$$|G_{\mathcal{G}_n}^{T_y}(y, y) - \tfrac{d}{d-1}| = O(\log^{-5} n).$$

All in all, we get that the first term of the RHS of (3.45) is $O(\log^{-5} n)$.
One applies to the second term of the RHS of (3.45) the same reasoning as that for (3.38) and (3.39). In particular, since $\mathcal{G}_n$ is a good graph, (3.11) yields $\max_{x,y\in V_n} |G_{\mathcal{G}_n}(x, y)| \leq K_5$, an upper bound easier to use than $\max_{z\in A} |\psi_{\mathcal{G}_n}(z)| \leq \log^{2/3} n$. □

3.5 Exploration of $\psi_{\mathcal{G}_n}$ around a vertex

3.5.1 Successful exploration

In this section, we prove that with $\mathbb{P}_{ann}$-probability arbitrarily close to $\eta(h)$ as $n \to +\infty$, $|\mathcal{C}_x^{\mathcal{G}_n,h}| \geq n^{1/2} \log^{-\kappa-6} n$, $\mathcal{C}_x^{\mathcal{G}_n,h}$ being the connected component of a given vertex x in $E_{\psi_{\mathcal{G}_n}}^{\geq h}$ (Proposition 3.5.1), and κ the constant defined in (3.33).
To do so, we explore a tree-like neighborhood T_x of x in $E_{\psi_{\mathcal{G}_n}}^{\geq h}$. We reveal T_x generation by generation, and couple it with a realization of $\mathcal{C}_\circ^{h+\log^{-1} n} \subseteq \mathbb{T}_d$ that is independent of the pairing of the half-edges of $\mathcal{G}_n$. By Proposition 3.2.4, a realization of $\psi_{\mathcal{G}_n}$ is given by a recursive construction with the same normal variables as those of the realization of $\varphi_{\mathbb{T}_d}$. When:

- that realization of $\mathcal{C}_\circ^{h+\log^{-1} n}$ is infinite (which happens with probability $\eta(h + \log^{-1} n) \simeq \eta(h)$), and
- the conditions of Proposition 3.4.1 hold for each vertex of T_x, until a generation at which $|\partial T_x| \geq n^{1/2} \log^{-\kappa-6} n$ (which happens with probability $1 - o(1)$, mainly because we have a probability $o(1)$ to create a cycle when pairing $o(\sqrt{n})$ half-edges),

we can apply Proposition 3.4.1 to bound the difference between $\varphi_{\mathbb{T}_d}$ and $\psi_{\mathcal{G}_n}$ by $\log^{-1} n$, ensuring that $T_x \subseteq \mathcal{C}_x^{\mathcal{G}_n,h}$, the connected component of x in $E_{\psi_{\mathcal{G}_n}}^{\geq h}$.

The exploration. Fix $x \in V_n$. Let

$$b_n := (d-1)^{-a_n} \log^{-6} n, \tag{3.46}$$

where we recall the definition of a_n from (3.33). Let $(\xi_y)_{y\in\mathbb{T}_d}$ be a family of independent variables, each of law $\mathcal{N}(0, 1)$, independent of the pairing of the half-edges in $\mathcal{G}_n$. Define the GFF $\varphi_{\mathbb{T}_d}$ as in Proposition 3.3.1.
At every step of the exploration, T_x will be a tree rooted at x, $\mathfrak{T}_x$ its respective counterpart in $\mathbb{T}_d$, rooted at $\circ$, and Φ an isomorphism from T_x to $\mathfrak{T}_x$. At step k, we will reveal the k-th generation of T_x and $\mathfrak{T}_x$.
Precisely, the **exploration from** x consists of the following steps:

- at step 0, $T_x = \{x\}$ and $\mathfrak{T}_x = \{\circ\}$. Reveal the pairings of the half-edges of $B_{\mathcal{G}_n}(x, a_n)$. Stop the exploration if $\mathrm{tx}(B_{\mathcal{G}_n}(x, a_n)) > 0$ or if $\varphi_{\mathbb{T}_d}(\circ) < h + \log^{-1} n$.

- at step $k \geq 1$, reveal the edges of $B_{\mathcal{G}_n}(T_x, a_n + 1)$ that were not known at step $k-1$. Let O_{k-1} be the set of the vertices of T_x of height $k-1$. Stop the exploration if at least one of the following conditions holds:

  C1 a cycle appears in $B_{\mathcal{G}_n}(T_x, a_n + 1)$,

  C2 $|O_{k-1}| \geq n^{1/2} b_n$,

  C3 $O_{k-1} = \emptyset$ (i.e. no vertex was added to T_x during the $(k-1)$-th step),

  C4 $k > \log_{\lambda_h} n$.

  Else, denote $x_{k,1}, x_{k,2}, \ldots, x_{k,m}$ the neighbours (in $\mathcal{G}_n$) of vertices of O_{k-1} that are not in O_{k-2}, for some $m \in \mathbb{N}$ (note that $m = (d-1)|O_{k-1}|$, each vertex of O_{k-1} having one neighbour in O_{k-2}, its parent, and $d-1$ other neighbours at distance k of $\circ$). Add to T_x the vertices $x_{k,i}$ of O_{k-1} such that $\varphi_{\mathbb{T}_d}(\Phi(x_{k,i})) \geq h + \log^{-1} n$. Add to $\mathfrak{T}_x$ the corresponding vertices $\Phi(x_{k,i})$.

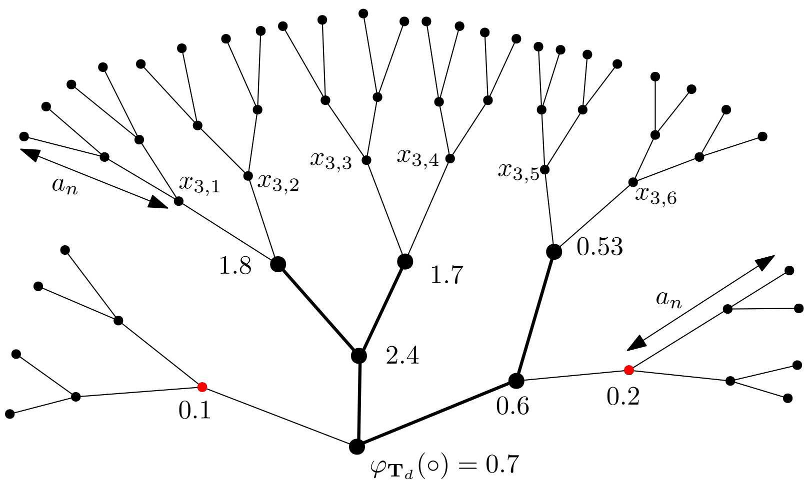

Figure 2. Illustration of the exploration with $k = 3$, $a_n = 2$, $h = 0.3$ and $n = 148$ (so that $\log^{-1} n \simeq 0.2$). Thick vertices and edges represent T_x after two steps. Red vertices have not been included in T_x because $\varphi_{\mathbb{T}_d}$ at their counterparts in $\mathbb{T}_d$ is below $h + \log^{-1} n$. Any number near a vertex v is $\varphi_{\mathbb{T}_d}(\Phi(v))$.

If the exploration is stopped at some step k, at which only C2 is met, say that it is **successful**. In this case, by Proposition 3.2.4, we can sample $\psi_{\mathcal{G}_n}$ as follows: we reveal the remaining pairings of half-edges in $\mathcal{G}_n$. We set $\psi_{\mathcal{G}_n}(x) = \varphi_{\mathbb{T}_d}(\circ)$. For all $k, i \geq 1$, if $A_{k,i} = \{x\} \cup \{x_{\ell,j} | (\ell, j) \prec (k,i)\}$ where $\prec$ is the lexicographical order on $\mathbb{N}^2$, let

$$\psi_{\mathcal{G}_n}(x_{k,i}) = \mathbb{E}^{\mathcal{G}_n}[\psi_{\mathcal{G}_n}(x_{k,i}) | \sigma(A_{k,i})] + \xi_{k,i} \sqrt{\mathrm{Var}(\psi_{\mathcal{G}_n}(x_{k,i}) | \sigma(A_{k,i}))}. \tag{3.47}$$

Note that the conditional expectation and variance of the RHS are $\sigma(A_{k,i})$-measurable random variables, so that this definition makes sense, even if $\psi_{\mathcal{G}_n}(x_{k,i})$ appears on both sides of the equation.

Let $\mathcal{S}(x) := \{$the exploration from x is successful$\}$. We prove the following:

Proposition 3.5.1.

$$\mathbb{P}_{ann}(\mathcal{S}(x) \cap \{T_x \subseteq \mathcal{C}_x^{\mathcal{G}_n,h}\}) \underset{n\to+\infty}{\longrightarrow} \eta(h). \tag{3.48}$$

Remark 3.5.2 (Exploration size). *Denote R_x the set of vertices* **seen** *during the exploration (i.e. such at least one of their half-edges has been paired). Note that for n large enough, for every $x \in V_n$, by C2, C4 and (3.46), T_x contains less than*

$$n^{1/2}(d-1)^{-a_n} \log^{-6} n \log_{\lambda_h} n \leq n^{1/2}(d-1)^{-a_n} \log^{-4} n$$

vertices, so that

$$|R_x| \leq n^{1/2}(d-1)^{-a_n} \log^{-4} n \times (1 + (d-1) + \ldots + (d-1)^{a_n+1}) \leq n^{1/2} \log^{-3} n.$$

In order to prove Proposition 3.5.1, we first show that $\mathcal{C}_\circ^{h+\log^{-1} n}$ either has an exponential growth at rate $> \sqrt{\lambda_h}$ with probability close to $\eta(h)$, or dies out before reaching height $\log_{\lambda_h} n$ with probability close to $1-\eta(h)$. Although the proof is slightly technical, it relies merely on Proposition 3.3.4 and on the continuity of the maps $h' \mapsto \lambda_{h'}$ and $h' \mapsto \eta(h')$.

Lemma 3.5.3. *Let $\mathcal{F}_k^{(n)} := \{n^{1/2}b_n \leq |\mathcal{Z}_k^{h+\log^{-1} n}| \leq dn^{1/2}b_n\}$ and $\mathcal{F'}_k^{(n)} := \{\mathcal{Z}_{k-1}^{h+\log^{-1} n} = \emptyset\}$ for $k \geq 1$. Let $k_0 := \inf\{k \geq 1, \mathcal{F}_k^{(n)}$ or $\mathcal{F'}_k^{(n)}$ happens$\}$, $\mathcal{F}^{*(n)} := \{k_0 \leq \log_{\lambda_h} n\} \cap \mathcal{F}_{k_0}^{(n)}$ and $\mathcal{F'}^{*(n)} := \{k_0 \leq \log_{\lambda_h} n\} \cap \mathcal{F'}_{k_0}^{(n)}$. Then, as $n \to +\infty$:*

$$\mathbb{P}_{ann}(\mathcal{F}^{*(n)}) \to \eta(h) \text{ and } \mathbb{P}_{ann}(\mathcal{F'}^{*(n)}) \to 1-\eta(h). \tag{3.49}$$

Proof. Remark first that by construction, $\mathbb{P}_{ann}$ acts like $\mathbb{P}_{\mathbb{T}_d}$ on events that only depend on $\varphi_{\mathbb{T}_d}$. Note that for every $n \geq 1$, $\mathcal{F'}^{*(n)} \cap \mathcal{F}^{*(n)} = \emptyset$, implying $\mathbb{P}_{ann}(\mathcal{F'}^{*(n)}) + \mathbb{P}_{ann}(\mathcal{F}^{*(n)}) \leq 1$. Hence it is enough to prove that

$$\liminf_{n\to+\infty} \mathbb{P}_{ann}(\mathcal{F}^{*(n)}) \geq \eta(h) \tag{3.50}$$

and

$$\liminf_{n\to+\infty} \mathbb{P}_{ann}(\mathcal{F'}^{*(n)}) \geq 1-\eta(h) \tag{3.51}$$

Let $\varepsilon \in (0, \eta(h))$. Let $\delta > 0$ be such that $|\eta(h+\delta) - \eta(h)| \leq \varepsilon$ and $\log_{\lambda_{h+\delta}} > 9\log_{\lambda_h}/10$. The map $h' \to \eta(h')$ is continuous on $\mathbb{R} \setminus \{h_\star\}$ by Theorem 3.1 of [3] and the map $h' \to \lambda_{h'}$ is an homeomorphism from $(-\infty, h_\star)$ to $(1, d-1)$ by Proposition 3.3.4, hence such δ exists. It is clear that

$$\begin{aligned}\liminf_{n\to+\infty} \mathbb{P}_{ann}\left(\exists k \leq \log_{\lambda_h} n, |\mathcal{Z}_k^{h+\log^{-1} n}| > n^{1/2}b_n\right) &\geq \liminf_{n\to+\infty} \mathbb{P}_{ann}\left(\exists k \leq \log_{\lambda_h} n, |\mathcal{Z}_k^{h+\delta}| > n^{1/2}b_n\right)\\ &\geq \liminf_{n\to+\infty} \mathbb{P}_{ann}\left(|\mathcal{Z}_{\lfloor\log_{\lambda_h} n\rfloor}^{h+\delta}| > n^{1/2}b_n\right)\\ &\geq \liminf_{n\to+\infty} \mathbb{P}_{ann}\left(|\mathcal{Z}_{\lfloor\log_{\lambda_h} n\rfloor}^{h+\delta}| > n^{9/10}/\log^2_{\lambda_h} n\right),\end{aligned}$$

hence by the first equation of Proposition 3.3.4 applied to $h+\delta$,

$$\liminf_{n\to+\infty} \mathbb{P}_{ann}\left(\exists k \le \log_{\lambda_h} n, |\mathcal{Z}_k^{h+\log^{-1} n}| > n^{1/2} b_n\right) \ge \eta(h+\delta) \ge \eta(h) - \varepsilon. \tag{3.52}$$

Since each vertex of $\mathbb{T}_d$ has at most d children, we have $|\mathcal{Z}_k^{h+\log^{-1} n}| \le d|\mathcal{Z}_{k-1}^{h+\log^{-1} n}|$ deterministically for all $k \ge 1$. Hence, letting $k' := \inf\{k \ge 0, |\mathcal{Z}_k^{h+\log^{-1} n}| \ge n^{1/2} b_n\}$ when this set is non-empty, $\mathcal{F}_{k'}^{(n)}$ holds, so that $\{\exists k \le \log_{\lambda_h} n, |\mathcal{Z}_k^{h+\log^{-1} n}| > n^{1/2} b_n\} \subseteq \cup_{k \le \log_{\lambda_h} n} \mathcal{F}_k^{(n)}$. Thus

$$\liminf_{n\to+\infty} \mathbb{P}_{ann}(\mathcal{F}^{*(n)}) \ge \liminf_{n\to+\infty} \mathbb{P}_{ann}\left(\cup_{k \le \log_{\lambda_h} n} \mathcal{F}_k^{(n)}\right) \ge \eta(h) - \varepsilon$$

by (3.52), and this shows (3.50).
For $n \ge e^{1/\delta}$, $\mathcal{C}_\circ^{h+\delta} \subseteq \mathcal{C}_\circ^{h+\log^{-1} n} \subseteq \mathcal{C}_\circ^h$. Note that for $n \ge 1$,

$$\begin{aligned}
&\mathbb{P}_{ann}(\mathcal{F}'^{*(n)}) \\
&\ge \mathbb{P}_{ann}\Big(\mathcal{Z}_{\lfloor \log_{\lambda_h} n \rfloor - 1}^{h+\log^{-1} n} = \emptyset\Big) - \mathbb{P}_{ann}\Big(\{\mathcal{Z}_{\lfloor \log_{\lambda_h} n \rfloor - 1}^{h+\log^{-1} n} = \emptyset\} \cap \{\exists k \ge 1, |\mathcal{Z}_k^{h+\log^{-1} n}| \ge n^{1/2} b_n\}\Big) \\
&\ge \mathbb{P}_{ann}\Big(\mathcal{Z}_{\lfloor \log_{\lambda_h} n \rfloor - 1}^{h} = \emptyset\Big) - \mathbb{P}_{ann}\left(\{|\mathcal{C}_\circ^{h+\log^{-1} n}| < +\infty\} \cap \{\exists k \ge 1, |\mathcal{Z}_k^{h+\log^{-1} n}| \ge n^{1/2} b_n\}\right) \\
&\ge \mathbb{P}_{ann}\Big(\mathcal{Z}_{\lfloor \log_{\lambda_h} n \rfloor - 1}^{h} = \emptyset\Big) - \mathbb{P}_{ann}\left(|\mathcal{C}_\circ^{h+\log^{-1} n}| < +\infty \mid \exists k \ge 1, |\mathcal{Z}_k^{h+\log^{-1} n}| \ge n^{1/2} b_n\right).
\end{aligned}$$

The first term of the RHS converges to $1-\eta(h)$ as $n \to +\infty$. For any $k \ge 1$ and for any $v \in B_{\mathbb{T}_d}(\circ, k)$, denoting T_v the possible subtree from v in $\mathcal{C}_\circ^{h+\log^{-1} n}$ (if $v \in \mathcal{Z}_k^{h+\log^{-1} n}$) and $\mathcal{C}_\circ(h,\delta)$ the connected component of $\circ$ in $(\{\circ\} \cup E_{\varphi_{\mathbb{T}_d}}^{\ge h+\delta}) \cap \mathbb{T}_d^+$,

$$\mathbb{P}_{ann}(|T_v| < +\infty \,|\, v \in \mathcal{Z}_k^{h+\log^{-1} n}) \le \mathbb{P}_h^{\mathbb{T}_d}(|\mathcal{C}_\circ(h,\delta)| < +\infty)$$

by Lemma 3.3.3, independently of the other vertices of $\mathcal{Z}_k^{h+\log^{-1} n}$. Thus,

$$\mathbb{P}_{ann}(|\mathcal{C}_\circ^h| < +\infty \mid \exists k \ge 1, |\mathcal{Z}_k^h| \ge n^{1/2} b_n) \le \mathbb{P}_h^{\mathbb{T}_d}(|\mathcal{C}_\circ(h,\delta)| < +\infty)^{n^{1/2} b_n}.$$

Therefore, it only remains to prove that $\mathbb{P}_h^{\mathbb{T}_d}(|\mathcal{C}_\circ(h,\delta)| < +\infty) < 1$. By Lemma 3.3.3, the map $a \mapsto \mathbb{P}_a^{\mathbb{T}_d}(|\mathcal{C}_\circ(h,\delta)| = +\infty)$ is non-decreasing. By (3.23),

$$\int_{h+\delta}^{+\infty} \mathbb{P}_a^{\mathbb{T}_d}(|\mathcal{C}_\circ(h,\delta)| = +\infty) \nu(da) > 0,$$

where we recall that ν is the density of $\varphi_{\mathbb{T}_d}(\circ)$. Hence for some $a_1 \ge h+\delta$, for every $a \ge a_1$, $\mathbb{P}_a^{\mathbb{T}_d}(|\mathcal{C}_\circ(h,\delta)| = +\infty) \ge \mathbb{P}_{a_1}^{\mathbb{T}_d}(|\mathcal{C}_\circ(h,\delta)| = +\infty) > 0$. And, there exists $p > 0$ such that

$$\mathbb{P}_h^{\mathbb{T}_d}(\circ \text{ has one child } z \text{ such that } \varphi_{\mathbb{T}_d}(z) \ge a_1) \ge p.$$

Therefore, $\mathbb{P}_h^{\mathbb{T}_d}(|\mathcal{C}_\circ(h,\delta)| = +\infty) \ge p\,\mathbb{P}_{a_1}^{\mathbb{T}_d}(|\mathcal{C}_\circ(h,\delta)| = +\infty) > 0$, and $\mathbb{P}_h^{\mathbb{T}_d}(|\mathcal{C}_\circ(h,\delta)| < +\infty) < 1$. (3.51) follows. □

Proof of Proposition 3.5.1. We first establish that C1 happens with $\mathbb{P}_{ann}$-probability $o(1)$. Then, if there is no cycle in $B_{\mathcal{G}_n}(T_x, a_n)$, we can apply Proposition 3.4.1, to bound the difference between $\varphi_{\mathbb{T}_d}$ and $\psi_{\mathcal{G}_n}$.

By Remark 3.5.2, at most $dn^{1/2}\log^{-3} n$ matchings of half-edges are performed during the exploration. By (3.14) with $k = m_0 = 1$, $m_E = 0$ and $m \leq dn^{1/2}\log^{-3} n$, the probability to create at least one cycle during these matchings is less than $\log^{-1} n$ for large enough n. Therefore,

$$\mathbb{P}_{ann}(\text{C1 happens}) \to 0. \tag{3.53}$$

Note that if C1 does not happen, then on $\mathcal{F}^{*(n)}$, (resp. $\mathcal{F'}^{*(n)}$), C3 (resp. C2) is satisfied, but not C4. Moreover, on $\mathcal{F}^{*(n)}$, (resp. $\mathcal{F'}^{*(n)}$), C2 (resp. C3) does not hold, so that

$$\mathbb{P}_{ann}(\mathcal{S}(x)) \geq \mathbb{P}_{ann}(\mathcal{F}^{*(n)}) - \mathbb{P}_{ann}(\text{C1 happens})$$

and, since $\mathcal{F}^{*(n)} \cap \mathcal{F'}^{*(n)} = \emptyset$:

$$\mathbb{P}_{ann}(\mathcal{S}(x)) \leq \mathbb{P}_{ann}((\mathcal{F}^{*(n)})^c) + \mathbb{P}_{ann}(\text{C1 happens}) \leq 1 - \mathbb{P}_{ann}(\mathcal{F'}^{*(n)}) + \mathbb{P}_{ann}(\text{C1 happens}).$$

Thus, by (3.49) and (3.53),

$$\mathbb{P}_{ann}(\mathcal{S}(x)) \to \eta(h). \tag{3.54}$$

Now, we compare $\psi_{\mathcal{G}_n}$ with $\varphi_{\mathbb{T}_d}$ (supposing that the exploration is over and that $\mathcal{S}(x)$ holds). Note that by C4, T_x has a maximal height $\log_{\lambda_h} n$ so that by the triangle inequality,

$$\{T_x \not\subseteq \mathcal{C}_x^{\mathcal{G}_n,h}\} \subseteq \{\exists y \in T_x,\, |\psi_{\mathcal{G}_n}(y) - \varphi_{\mathbb{T}_d}(\Phi(y))| \geq \log^{-1} n\} \subseteq \cup_{y\in T_x}\mathcal{E}(y),$$

where $\mathcal{E}(x) := \{|\psi_{\mathcal{G}_n}(x) - \varphi_{\mathbb{T}_d}(\Phi(x))| \geq \log^{-3} n\}$ and

$$\mathcal{E}(y) := \{|\psi_{\mathcal{G}_n}(y) - \varphi_{\mathbb{T}_d}(\Phi(y))| \geq |\psi_{\mathcal{G}_n}(\overline{y}) - \varphi_{\mathbb{T}_d}(\overline{\Phi(y)})| + 2\log^{-3} n\} \text{ for } y \neq x.$$

Suppose that $\mathcal{G}_n$ is a good graph. For $x_{k,i} \in T_x \setminus \{x\}$, we can apply Proposition 3.4.1 on the event $\mathcal{E}_{k,i}^{(n)} := \{\max_{y'\in A_{k,i}} |\psi_{\mathcal{G}_n}(y')| < \log^{2/3} n\}$, (note that $|A_{k,i}| \leq n^{1/2}$ by Remark 3.5.2 and that $\text{tx}(B_{\mathcal{G}_n}(A_{k,i}, a_n)) = \text{tx}(A_{k,i})$ by C1). Writing $y = x_{k,i}$ and $\xi = \xi_{\Phi(x_{k,i})}$, we get for n large enough:

$$\begin{aligned}|\psi_{\mathcal{G}_n}(y) - \varphi_{\mathbb{T}_d}(\Phi(y))| \leq &\left|\frac{\psi_{\mathcal{G}_n}(\overline{y}) - \varphi_{\mathbb{T}_d}(\overline{\Phi(y)})}{d-1}\right| + \log^{-3} n \\ &+ \left|\left(\sqrt{\text{Var}^{\mathcal{G}_n}(\psi_{\mathcal{G}_n}(y)|\sigma(A))} - \sqrt{\frac{d-1}{d}}\right)\xi\right|\end{aligned}$$

and

$$\begin{aligned}\left|\sqrt{\text{Var}^{\mathcal{G}_n}(\psi_{\mathcal{G}_n}(y)|\sigma(A))} - \sqrt{\frac{d-1}{d}}\right| &\leq \left|\sqrt{\frac{d-1}{d} - \log^{-4} n} - \sqrt{\frac{d-1}{d}}\right| \\ &\leq \frac{11}{10}\sqrt{\frac{d-1}{d}}\frac{d}{2(d-1)}\log^{-4} n \\ &\leq \log^{-4} n.\end{aligned}$$

Let $\mathcal{E}'(y) := \mathcal{E}(y) \cap \mathcal{E}_{k,i}^{(n)}$. We have

$$\mathbb{P}^{\mathcal{G}_n}(\mathcal{E}'(y)) \leq \mathbb{P}(|\xi|\log^{-4} n \geq \log^{-3} n) \leq n^{-3} \tag{3.55}$$

by the exponential Markov inequality used as in the proof of Lemma 3.2.5. Moreover, by (3.8), if κ is large enough, we obtain in the same manner:

$$\mathbb{P}^{\mathcal{G}_n}(\mathcal{E}(x)) \leq \mathbb{P}^{\mathcal{G}_n}(|\xi_\circ| \log^{-4} n \geq \log^{-3} n) \leq n^{-3}. \tag{3.56}$$

By Remark 3.5.2, we have $|T_x| \leq n^{1/2}$. By (3.55), (3.56) and a union bound on $y \in T_x$, $\mathbb{P}^{\mathcal{G}_n}(\cup_{y \in T_x} \mathcal{E}'(y)) \leq n^{-5/2}$ with $\mathcal{E}'(x) := \mathcal{E}(x)$. And $\mathbb{P}^{\mathcal{G}_n}(\cup_{(k,i):x_{k,i} \in T_x} \mathcal{E}^{(n)}_{k,i}) \leq n^{-2}$ for large enough n, by Lemma 3.2.5.
Thus for n large enough, if $\mathcal{G}_n$ is a good graph,

$$\mathbb{P}^{\mathcal{G}_n}(T_x \not\subseteq \mathcal{C}_x^{\mathcal{G}_n,h}) \leq \mathbb{P}^{\mathcal{G}_n}(\cup_{y \in T_x} \mathcal{E}(y)) \leq n^{-5/2} + n^{-2} \leq n^{-1},$$

so that by Proposition 3.2.1 and (3.54):

$$\mathbb{P}_{ann}(\mathcal{S}(x) \cap \{T_x \subseteq \mathcal{C}_x^{\mathcal{G}_n,h}\}) \to \eta(h).$$

□

3.5.2 Aborted exploration

For $x \in V_n$, the **lower exploration** is the exploration of Section 3.5.1, modified by replacing $h + \log^{-1} n$ by $h - \log^{-1} n$, so that we compare T_x and $\mathcal{C}_\circ^{h - \log^{-1} n}$. If it is stopped at some step k at which only C3 is met, say that it is **aborted**. Write
$\mathcal{A}(x) := \{\text{the lower exploration from } x \text{ is aborted}\} \cap \{\mathcal{C}_x^{\mathcal{G}_n,h} \subseteq T_x\}$.

Proposition 3.5.4.

$$\mathbb{P}_{ann}(\mathcal{A}(x)) \underset{n \to +\infty}{\longrightarrow} 1 - \eta(h). \tag{3.57}$$

The proof follows from a direct adaptation of Lemma 3.5.3 and Proposition 3.5.1. Note in particular that $\mathbb{P}_{ann}(\text{C1 happens}) = o(1)$, and that $\mathbb{P}_{ann}(\mathcal{A}(x)) = \mathbb{P}_{ann}(\mathcal{Z}_{k_0-1}^{h-\log^{-1} n} = \emptyset) + o(1) = 1 - \eta(h) + o(1)$.

3.6 Existence of a giant component

In Section 3.6.1, we prove that two vertices $x, y \in V_n$ are in the same connected component of $E_{\psi_{\mathcal{G}_n}}^{\geq h}$ with $\mathbb{P}_{ann}$-probability $\underset{n \to +\infty}{\longrightarrow} \eta(h)^2$ (Proposition 3.6.2). Then in Section 3.6.2, we use a second moment argument to get concentration and to show (3.2).

3.6.1 Connecting two successful explorations

Let us describe our strategy to establish Proposition 3.6.2. We perform explorations as in Section 3.5.1 from x and y. If they are both successful and do not meet (which happens with probability $\simeq \eta(h)^2$), we develop disjoint balls, denoted "joining balls", from ∂T_x to ∂T_y (Section 3.6.1). Each of them is rooted at a vertex of ∂T_x, hits ∂T_y at exactly one vertex, and has a "security radius" of depth a_n around its path from ∂T_x to ∂T_y (see Figure 3). The construction of the joining balls only depends on the structure of $\mathcal{G}_n$, and not on the values of $\psi_{\mathcal{G}_n}$. Then, we realize $\psi_{\mathcal{G}_n}$ on T_x, T_y and those balls (Section 3.6.1). If they are all disjoint and tree-like, once we have revealed $\psi_{\mathcal{G}_n}$ on T_x and T_y, this security radius allows us to apply Proposition 3.4.1 to approximate $\psi_{\mathcal{G}_n}$ on the paths from ∂T_x to ∂T_y by $\varphi_{\mathbb{T}_d}$.

Let us explain with a back-of-the-envelope computation how the joining balls allow to connect T_x and T_y in $E^{\geq h}_{\psi_{\mathcal{G}_n}}$. Since $|T_y| \simeq n^{1/2} b_n$ by C2, the probability that for a given $z \in \partial T_x$, exactly one of the vertices at distance $\lfloor \gamma \log_{d-1} \log n \rfloor$ (and no vertex at distance $< \lfloor \gamma \log_{d-1} \log n \rfloor$) from z is in ∂T_y is

$$\simeq \mathbb{P}(\mathrm{Bin}((d-1)^{\gamma \log_{d-1} \log n}, \tfrac{n^{1/2} b_n}{dn}) = 1) \simeq \log^{\gamma} n \times n^{-1/2} b_n.$$

And there are $\simeq n^{1/2} b_n$ vertices in ∂T_x, hence we can expect that that the number of joining balls is at least $\simeq n^{1/2} b_n \times \log^{\gamma} n \times n^{-1/2} b_n = b_n^2 \log^{\gamma} n$, provided that we can control some undesirable events (such as an intersection between balls, or a cycle in a ball). This is the purpose of Lemma 3.6.1.

Moreover, we know that for large $r \in \mathbb{N}$ and $v \in \partial B_{\mathbb{T}_d}(\circ, r)$, $\mathbb{P}^{\mathbb{T}_d}(v \in \mathcal{C}^h_{\circ})$ is of order $(\lambda_h/(d-1))^r$, by Proposition 3.3.8. Taking $r = \gamma \log_{d-1} \log n$, the probability that $E^{\geq h}_{\psi_{\mathcal{G}_n}}$ percolates from ∂T_x to ∂T_y through a given joining ball is $\simeq \log^{\gamma(\log_{d-1} \lambda_h - 1)} n$, if we can approximate $\psi_{\mathcal{G}_n}$ by $\varphi_{\mathbb{T}_d}$. For γ large enough w.r.t κ (recall (3.46) and (3.33), and recall that $\lambda_h > 1$),

$$b_n^2 \log^{\gamma} n \times \log^{\gamma(\log_{d-1} \lambda_h - 1)} n \geq \log^{-2\kappa - 13} n \times \log^{\gamma \log_{d-1} \lambda_h} n >> 1,$$

so that with high probability, $E^{\geq h}_{\psi_{\mathcal{G}_n}}$ will percolate through at least one joining ball from ∂T_x to ∂T_y.

The joint exploration

For $x, y \in V_n$, write $x \overset{h}{\leftrightarrow} y$ if $y \in \mathcal{C}^{\mathcal{G}_n, h}_x$. Let $(\xi_{z,v})_{z \in \{x,y\}, v \in \mathbb{T}_d}$ be an array of i.i.d. standard normal variables independent from everything else. Define the **joint exploration from x and y** as the exploration from x (with the $(\xi_{x,v})$'s), then the exploration from y (with the $(\xi_{y,v})$'s), as in Section 3.5.1, with the additional condition

C5 the exploration is stopped as soon as $R_x \cap R_y \neq \emptyset$,

where R_x (resp. R_y) is the set of vertices seen during the exploration from x (resp. from y), as defined in Remark 3.5.2. Note that the families $(\xi_{x,v})_{v \in \mathbb{T}_d}$ and $(\xi_{x,v})_{v \in \mathbb{T}_d}$ generate two independent copies of $\varphi_{\mathbb{T}_d}$.

If both explorations are successful and C5 does not happen (denote $\mathcal{S}(x, y)$ this event), we add the following steps to the joint exploration. Let $\gamma > 0$ and

$$a'_n := \lfloor \gamma \log_{d-1} \log n \rfloor. \tag{3.58}$$

Denote $z_1, \dots, z_{|\partial T_x|}$ the vertices of ∂T_x. For $j = 1, 2, \dots, |\partial T_x|$ successively, build $B^*(z_j, a'_n)$ the subgraph of $\mathcal{G}_n$ obtained as follows (see Figure 3 for an illustration). Write

$$B^*_j := \cup_{j'<j} B^*(z_{j'}, a'_n) \text{ and } Q_j := R_x \cup R_y \cup B^*_j, \tag{3.59}$$

so that Q_j is the set of vertices seen in the exploration before the construction of $B^*(z_j, a'_n)$.
Let initially $B^*(z_j, a'_n)$ be the subtree from z_j of height a_n in the tree $B_{\mathcal{G}_n}(T_x, a_n)$ (in blue in Figure 3). For $k \le a'_n$, write $B^*(z_j, k) := B^*(z_j, a'_n) \cap B_{\mathcal{G}_n}(z_j, k)$. If $B^*(z_j, a_n) \cap B^*_j \neq \emptyset$, say that j is **spoiled**, and the construction of $B^*(z_j, a'_n)$ stops.
Else, for $k = a_n, a_n + 1, \dots, a'_n - 2a_n - 2$ successively, while $\text{tx}(B^*(z_j, k) \cup Q_j) = \text{tx}(Q_j)$ (i.e. no cycle has been discovered) and $B^*(z_j, k) \cap B_{\mathcal{G}_n}(T_y, a_n) = \emptyset$, add to $B^*(z_j, a'_n)$ the neighbours of $B^*(z_j, k)$ and the corresponding edges (in red in Figure 3). If for some $k \in \{a_n, \dots, a'_n - 2a_n - 2\}$, $\text{tx}(B^*(z_j, k) \cup Q_j) > \text{tx}(Q_j)$ (i.e. at least one cycle is discovered) or $B^*(z_j, k) \cap B_{\mathcal{G}_n}(T_y, a_n) \neq \emptyset$, the construction of $B^*(z_j, a'_n)$ stops.
If the construction has not been stopped for some $k < a'_n - 2a_n - 1$, add the neighbours of $B^*(z_j, a'_n - 2a_n - 1)$ to $B^*(z_j, a'_n)$ (also in red in Figure 3). If

$$|B^*(z_j, a'_n - 2a_n) \cap B_{\mathcal{G}_n}(T_y, a_n)| \neq 1,$$

the construction of $B^*(z_j, a'_n)$ stops.
Else, let $v_j(0)$ be the unique vertex of $B^*(z_j, a'_n - 2a_n) \cap B_{\mathcal{G}_n}(T_y, a_n)$. If

$$\text{tx}((B^*(z_j, a'_n - 2a_n) \cup Q_j) \setminus \{v_j(0)\}) > \text{tx}(Q_j),$$

the construction of $B^*(z_j, a'_n)$ stops.
Else, for $k = a'_n - 2a_n, \dots, a'_n - 1$ successively, while $\text{tx}(B^*(z_j, k) \cup Q_j) = \text{tx}(B^*(z_j, a'_n - 2a_n) \cup Q_j)$, add the neighbours of $B^*(z_j, k)$ to $B^*(z_j, a'_n)$ (in green in Figure 3). Then, the construction of $B^*(z_j, a'_n)$ is completed. In this case only, and if

$$\text{tx}(B^*(z_j, a'_n) \cup Q_j) = \text{tx}(B^*(z_j, a'_n - 2a_n) \cup Q_j),$$

say that $B^*(z_j, a'_n)$ is a **joining ball**. In other words, we obtain a joining ball if, revealing the offspring up to generation a'_n of z_j, the $(a'_n - 2a_n)$ offspring of z_j intersects ∂T_y at a unique vertex $v_j(0)$, and no cycle is discovered in the whole construction (except when $B^*(z_j, a'_n)$ reaches $\partial B_{\mathcal{G}_n}(T_y, a_n)$ at $v_j(0)$, if T_x and T_y were already connected in Q_j by $B^*(z_{j'}, a'_n)$ for some $j' < j$).
Write $J := \{j \le |\partial T_x|, B^*(z_j, a'_n) \text{ is a joining ball}\}$ and $\mathcal{S}'(x, y) := \mathcal{S}(x, y) \cap \{|J| \ge \log^{\gamma - 3\kappa - 18} n\}$ ($\mathcal{S}(x, y)$ was define above (3.58)).

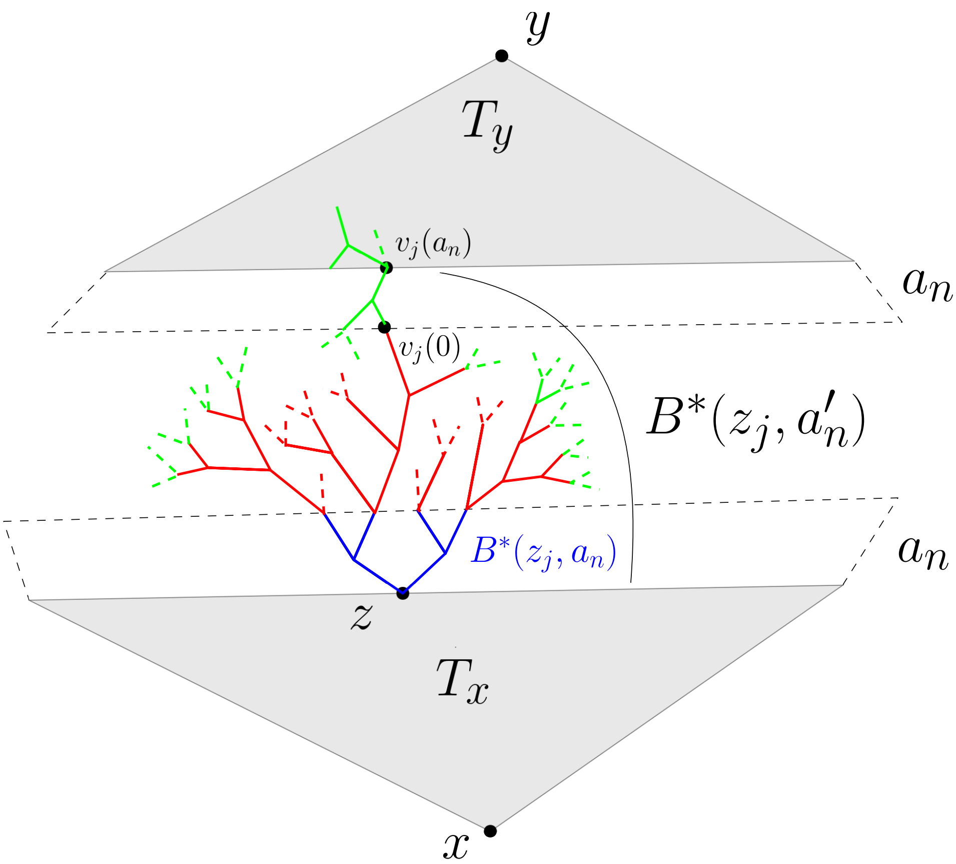

Figure 3. Illustration of a joining ball $B^*(z_j, a'_n)$. Here $a_n = 2$ and $a'_n = 9$. Dashed lines represent subtrees that have not been fully pictured. The blue tree has total height a_n, the red trees $a'_n - 3a_n$, and the green trees $2a_n$.

Lemma 3.6.1. *Fix $\gamma > 3\kappa + 18$.*

$$\mathbb{P}_{ann}(\mathcal{S}'(x,y)) \underset{n\to+\infty}{\longrightarrow} \eta(h)^2.$$

Proof. Denote $\mathcal{F}^{*(n)}(x)$ (resp. $\mathcal{F}^{*(n)}(y)$) the event $\mathcal{F}^{*(n)}$ for x (resp. y). Remark that the realization of $\mathcal{F}^{*(n)}(x)$ (resp. $\mathcal{F}^{*(n)}(y)$) only depends on the version of $\varphi_{\mathbb{T}_d}$ defined by $(\xi_{x,v})_{v\in\mathbb{T}_d}$ (resp. by $(\xi_{y,v})_{v\in\mathbb{T}_d}$) and not on the pairings of $\mathcal{G}_n$. Hence, $\mathcal{F}^{*(n)}(x)$ and $\mathcal{F}^{*(n)}(y)$ are independent. As in the proof of Lemma 3.5.3, we get that

$$\mathbb{P}_{ann}(\mathcal{F}^{*(n)}(x) \cap \mathcal{F}^{*(n)}(y)) = \mathbb{P}_{ann}(\mathcal{F}^{*(n)}(x))\mathbb{P}_{ann}(\mathcal{F}^{*(n)}(y)) \to \eta(h)^2$$

and $\mathbb{P}_{ann}(\mathcal{F}'^{*(n)}(x) \cup \mathcal{F}'^{*(n)}(y)) \to 1 - \eta(h)^2$.
Moreover, $\mathbb{P}_{ann}(\text{C1 or C5 happens}) \to 0$. Indeed, by Remark 3.5.2, less than $2dn^{1/2}\log^{-3} n$ half-edges are revealed during the explorations from x and y, which allows to control C5 as we did for C1 in (3.53). Thus,

$$\limsup_{n\to+\infty} |\mathbb{P}_{ann}(\mathcal{S}'(x,y)) - \eta(h)^2| \leq \limsup_{n\to+\infty} \mathbb{P}_{ann}(\mathcal{S}(x,y) \cap \{|J| < \log^{\gamma-3\kappa-18} n\})$$

and it remains to prove that

$$\limsup_{n\to+\infty} \mathbb{P}_{ann}(\mathcal{S}(x,y)\cap\{|J|<\log^{\gamma-3\kappa-18} n\})=0. \tag{3.60}$$

We proceed in two steps: in step 1, we control the number of spoiled vertices, and the number of vertices of $\partial B_{\mathcal{G}_n}(T_y,a_n)$ that are hit when building the $B^*(z_j,a'_n)$'s (if a large proportion of those vertices are in Q_j, then it significantly affects the probability that $B^*(z_j,a'_n)$ is a joining ball). In step 2, we estimate the probability that for a given j, $B^*(z_j,a'_n)$ is a joining ball, provided that the bounds of step 1 hold. This gives a binomial lower bound for $|J|$.

Step 1. Note that by Remark 3.5.2 and (3.33), $|\partial B_{\mathcal{G}_n}(T_x,a_n)|+|\partial B_{\mathcal{G}_n}(T_y,a_n)|\le 2n^{1/2}\log^{-1} n$. Note also that for every $j\le|\partial T_x|$, $B^*(z_j,a'_n)$ contains less than $(d-1)^{a'_n}\le\log^{\gamma} n$ half-edges. Hence:

$$\text{at every moment of the exploration, less than } n^{1/2}\log^{\gamma} n \text{ half-edges have been seen.} \tag{3.61}$$

Let

$$B^* := \cup_{j\le|\partial T_x|} B^*(z_j,a'_n). \tag{3.62}$$

To reveal the edges of B^*, one proceeds to at most $n^{1/2}\log^{\gamma} n$ pairings of half-edges by (3.61). Any pairing that results in an edge e between some $B^*(z_j,k)$ and $B_{\mathcal{G}_n}(T_y,a_n)$ then leads to at most

$$1+(d-1)+\ldots+(d-1)^{2a_n}\le 3(d-1)^{2a_n}\le\log^{2\kappa+1} n\le\log^{\gamma-1} n$$

vertices of $B^*(z_j,a'_n)\cap B_{\mathcal{G}_n}(T_y,a_n)$, since the construction of $B^*(z_j,a'_n)$ stops if such an edge happens at distance less than a'_n-2a_n of z_j (and recall that we choose $\gamma>3\kappa+18>2\kappa+2$). Thus, by (3.15) with $k=\lfloor\log^{2\gamma+1} n\rfloor$, $m<n^{1/2}\log^{\gamma} n$ and $m_1+m_0+m_E<n^{1/2}\log^{\gamma} n$ (due to 3.61), for n large enough:

$$\mathbb{P}_{ann}(\mathcal{S}(x,y)\cap\{|B_{\mathcal{G}_n}(T_y,a_n)\cap B^*|\ge\log^{3\gamma} n\})\le 0.99^{\log^2 n}\le n^{-3}. \tag{3.63}$$

Let N be the total number of spoiled vertices. By (3.15) with the same parameters,

$$\mathbb{P}_{ann}(\mathcal{S}(x,y)\cap\{N\ge\log^{3\gamma} n\})\le n^{-3}. \tag{3.64}$$

Step 2. Recall the definition of B^*_j from (3.59). For $j\le m$, denote

$$\mathcal{S}_j := \mathcal{S}(x,y)\cap\{|B_{\mathcal{G}_n}(T_y,a_n)\cap B^*_j|\le\log^{3\gamma} n\}\cap\{z_j \text{ is not spoiled}\}. \tag{3.65}$$

Suppose that for every $j\ge 1$,

$$\mathbb{P}_{ann}(B^*(z_j,a'_n) \text{ is a joining ball} \mid \mathcal{S}_j)\ge n^{-1/2}\log^{\gamma-2\kappa-10} n. \tag{3.66}$$

On $\mathcal{E} := \{|B_{\mathcal{G}_n}(T_y,a_n)\cap B^*|<\log^{3\gamma} n\}\cap\{N<\log^{3\gamma} n\}$, the number of j's such that $\mathcal{S}_j$ holds is at least

$$|\partial T_x| - \log^{3\gamma} n \geq n^{1/2} \log^{-\kappa-7} n$$

by (C2) and (3.46). Thus, if $Z \sim \text{Bin}\left(\lfloor n^{1/2} \log^{-\kappa-7} n\rfloor, n^{-1/2}\log^{\gamma-2\kappa-10} n\right)$,

$$\mathbb{P}_{ann}\left(\mathcal{S}(x,y) \cap \{|J| \leq \log^{\gamma-3\kappa-18} n\}\right) \leq \mathbb{P}(Z \leq \log^{\gamma-3\kappa-18} n) + \mathbb{P}_{ann}(\mathcal{S}(x,y) \cap \mathcal{E}^c).$$

For large enough n, $\mathbb{P}_{ann}(\mathcal{S}(x,y) \cap \mathcal{E}^c) = o(n^{-2})$ by (3.63) and (3.64). Moreover, one checks easily (using $\gamma > 3\kappa + 18$ and (3.12)) that for n large enough:

$$\max_{0 \leq k \leq \lfloor \log^{\gamma-3\kappa-18} n\rfloor} \mathbb{P}(Z = k) \leq 1/n.$$

This yields (3.60). Hence, it only remains to prove (3.66).

Remark that $\mathbb{P}_{ann}(B^*(z_j, a'_n)$ is a joining ball $|\, \mathcal{S}_j) \geq p_1 p_2 p_3$ where:

- $p_1 := \mathbb{P}_{ann}(\mathcal{E}_1 | \mathcal{S}_j)$ and $\mathcal{E}_1 := \mathcal{S}_j \cap \{$no cycle is created and no connection to Q_j is made when revealing $B^*(z_j, a'_n - 2a_n - 1)\}$,
- $p_2 := \mathbb{P}_{ann}(\mathcal{E}_2|\mathcal{E}_1)$ where $\mathcal{E}_2 := \mathcal{E}_1 \cap \{$exactly one edge connects $B^*(z_j, a'_n - 2a_n - 1)$ and $D := \partial B_{\mathcal{G}_n}(T_y, a_n) \backslash \{B_{\mathcal{G}_n}(B_{\mathcal{G}_n}(T_y, a_n) \cap B^*_j, 2a_n)\}\} \cap \{$no cycle is created and no connection to $\partial B_{\mathcal{G}_n}(T_y, a_n) \cup B^*_j$ is made when revealing the other edges of $B^*(z_j, a'_n - 2a_n)\}$,
- $p_3 := \mathbb{P}_{ann}(\mathcal{E}_3|\mathcal{E}_2)$ where $\mathcal{E}_3 := \mathcal{E}_2 \cap \{$no cycle is created and no connection to $\partial B_{\mathcal{G}_n}(T_y, a_n) \cup B^*_j$ is made when revealing the remaining edges of $B^*(z_j, a'_n)\}$.

This definition of D guarantees that $B^*(z_j, a'_n)$ will not intersect a previously realized joining ball when growing the subtree from $v_j(0)$ in $B_{\mathcal{G}_n}(T_y, a_n)$.
(3.14) with $k = 1$, $m_0, m \leq \log^\gamma n$ and $m_E, m_1 \leq n^{1/2} \log^\gamma n$ due to (3.61) yields for n large enough:

$$p_i \geq 1 - C(1) \log^\gamma n \, \frac{\max(n^{1/2} \log^\gamma n, 2 \log^\gamma n)}{n} \geq 1 - n^{-1/3} \tag{3.67}$$

for $i \in \{1, 3\}$. Therefore, $p_1 p_3 \geq 1/2$ for n large enough.
On $\mathcal{E}_1$, reveal the pairings of the half-edges of $\partial B^*(z_j, a'_n - 2a_n - 1)$ one by one. $\mathcal{E}_2$ holds if:

- one given half-edge is matched to a half-edge of D, which happens with probability at least $\frac{|D|}{dn}$, and
- each other half-edge is matched to a half-edge that had not been seen before (by (3.61), for each half-edge this happens with probability at least $1 - \frac{n^{1/2}\log^\gamma n}{dn - n^{1/2}\log^\gamma n} \geq 1 - \frac{n^{1/2}\log^\gamma n}{n}$).

Since $\partial B^*(z_j, a'_n - 2a_n - 1)$ has $(d-1)|\partial B^*(z_j, a'_n - 2a_n - 1)|$ unpaired half-edges,

$$p_2 \geq (d-1)|\partial B^*(z_j, a'_n - 2a_n - 1)| \frac{|D|}{dn} \left(1 - \frac{n^{1/2}\log^\gamma n}{n}\right)^{|\partial B^*(z_j, a'_n - 2a_n - 1)| - 1}$$

By (3.33) and (3.58), one checks easily that on $\mathcal{E}_1$,

$$\log^{\gamma-2\kappa-1} n \leq |\partial B^*(z_j, a'_n - 2a_n - 1)| \leq \log^\gamma n,$$

and that on $\mathcal{S}_j$ (defined in (3.65)),

$$|D| \geq \frac{|\partial B_{\mathcal{G}_n}(T_y, a_n)|}{2} \geq n^{1/2} \log^{-7} n$$

by (3.46) and C2. Hence for n large enough,

$$\begin{aligned} p_2 &\geq (d-1) \log^{\gamma - 2\kappa - 1} n \, \frac{n^{1/2} \log^{-7} n}{dn} \left(1 - \frac{n^{1/2} \log^{\gamma} n}{dn}\right)^{\log^{\gamma} n} \\ &\geq \frac{1}{2} n^{-1/2} \log^{\gamma - 2\kappa - 9} n. \end{aligned}$$

With (3.67), this entails

$$\mathbb{P}_{ann}(B^*(z_j, a_n') \text{ is a joining ball} \,|\, \mathcal{S}_j) \geq p_1 p_2 p_3 \geq n^{-1/2} \log^{\gamma - 2\kappa - 10}$$

for n large enough (uniformly on the realization of B_j^*), and thus (3.66), so that the proof of the Lemma is complete. □

The field $\psi_{\mathcal{G}_n}$ on the joint exploration

Suppose that we are on $\mathcal{S}'(x, y)$. By Proposition 3.2.4, we can realize $\psi_{\mathcal{G}_n}$ on T_x as in (3.47) with the $(\xi_{x,v})_{v \in \mathbb{T}_d}$. Then we can realize it in a similar way on T_y with the $(\xi_{y,v})_{v \in \mathbb{T}_d}$, letting recursively

$$\psi_{\mathcal{G}_n}(y_{k,i}) = \mathbb{E}^{\mathcal{G}_n}[\psi_{\mathcal{G}_n}(y_{k,i}) | \sigma(A_{k,i})] + \xi_{y, \Phi(y_{k,i})} \sqrt{\mathrm{Var}(\psi_{\mathcal{G}_n}(y_{k,i}) | \sigma(A_{k,i}))}$$

where $A_{k,i} := T_x \cup \{y_{\ell,j} | (\ell, j) \prec (k, i)\}$, $\prec$ being the lexicographical order on $\mathbb{N}^2$, and Φ is the isomorphism between T_y and $\mathfrak{T}_y$.
Recall that $J = \{j \geq 1,\, B^*(z_j, a_n') \text{ is a joining ball}\}$ and that for $j \in J$, we denote $v_j(0)$ the unique vertex of $B^*(z_j, a_n' - 2a_n) \cap B_{\mathcal{G}_n}(T_y, a_n)$. Since no cycle is discovered when revealing $B^*(z_j, a_n') \setminus B^*(z_j, a_n' - 2a_n)$, the intersection of $B^*(z_j, a_n' - a_n)$ and T_y is a unique vertex $v_j(a_n)$, which is in the a_n-offspring of $v_j(0)$ in the tree $B^*(z_j, a_n')$ rooted at z_j. Then we realize $\psi_{\mathcal{G}_n}$ on $B^*(z_j, a_n' - 2a_n)$ and on the shortest path P_j from $v(0)$ to $v(a_n)$ as in (3.47), via a family of i.i.d. $\mathcal{N}(0,1)$ random variables $(\xi_{j,k,i})_{k,i \geq 0}$. In the tree $T_j := B^*(z_j, a_n' - 2a_n) \cup P_j$ with root z_j, denoting $z_{j,k,i}$ the i-th vertex at generation k and

$$A_{j,k,i} := T_x \cup T_y \cup \{\cup_{j' < j} T_j'\} \cup \{y_{j,k',i'} \,|\, (k', i') \prec (k, i)\}$$

the set of vertices where $\psi_{\mathcal{G}_n}$ has already been revealed before $z_{j,k,i}$, we let

$$\psi_{\mathcal{G}_n}(z_{j,k,i}) = \mathbb{E}^{\mathcal{G}_n}[\psi_{\mathcal{G}_n}(z_{j,k,i}) | \sigma(A_{j,k,i})] + \xi_{j,k,i} \sqrt{\mathrm{Var}(\psi_{\mathcal{G}_n}(y_{j,k,i}) | \sigma(A_{j,k,i}))}. \quad (3.68)$$

Write $\mathcal{S}^*(x, y) \subseteq \mathcal{S}'(x, y)$ the event that there exists $j_0 \geq 1$ and a path from z_{j_0} to $v_{j_0}(a_n)$ such that $\psi_{\mathcal{G}_n}(v) \geq h$ for every vertex v of that path. In particular, on $\mathcal{S}^*(x, y)$, x and y are in the same connected component of $E_{\psi_{\mathcal{G}_n}}^{\geq h}$.
Recall the definitions of κ (3.33) and γ (Lemma 3.6.1).

Proposition 3.6.2. *If κ and γ/κ are large enough, then*

$$\mathbb{P}_{ann}(\mathcal{S}^*(x,y)) \underset{n\to+\infty}{\longrightarrow} \eta(h)^2.$$

Proof of Proposition 3.6.2. Let $\gamma > 3\kappa + 18$. By Lemma 3.6.1,

$$\limsup_{n\to+\infty} \mathbb{P}_{ann}(\mathcal{S}^*(x,y)) \le \lim_{n\to+\infty} \mathbb{P}_{ann}(\mathcal{S}'(x,y)) = \eta(h)^2.$$

Let $\mathcal{E}_n := \{\mathcal{G}_n \text{ is not a good graph}\} \cup \{\max_{z\in V_n} |\psi_{\mathcal{G}_n}(z)| \ge \log^{2/3} n\}$. By Proposition 3.2.1 and Lemma 3.2.5, $\mathbb{P}_{ann}(\mathcal{E}_n) \to 0$. Therefore, it is enough to show that

$$\limsup_{n\to+\infty} \mathbb{P}_{ann}(\mathcal{E}_n^c \cap (\mathcal{S}'(x,y) \setminus \mathcal{S}^*(x,y))\,) = 0.$$

By a straightforward adaptation of the reasoning below (3.54),

$$\lim_{n\to+\infty} \mathbb{P}_{ann}(\mathcal{E}_n^c \cap (\mathcal{S}'(x,y) \setminus \mathcal{S}''(x,y))\,) = 0,$$

where $\mathcal{S}''(x,y) := \mathcal{S}'(x,y) \cap \{\forall z \in T_x \cup T_y,\, \psi_{\mathcal{G}_n}(z) \ge h + (\log^{-1} n)/2\}$. Hence, we are left with proving that

$$\limsup_{n\to+\infty} \mathbb{P}_{ann}(\mathcal{E}_n^c \cap (\mathcal{S}''(x,y) \setminus \mathcal{S}^*(x,y))\,) = 0. \tag{3.69}$$

We use again a binomial argument. For $j \in J$ in increasing order, generate the GFF on T_j as in (3.68). Denote E_j the event that z_j and $v_j(a_n)$ are in the same connected component of $E^{\ge h}_{\psi_{\mathcal{G}_n}} \cap T_j$. Note that on $\mathcal{S}''(x,y)$, $T_x \subseteq \mathcal{C}_x^{\mathcal{G}_n,h}$ and $T_y \subseteq \mathcal{C}_y^{\mathcal{G}_n,h}$, so that $\mathcal{S}''(x,y) \cap (\cup_{j\in J} E_j) \subseteq \mathcal{S}^*(x,y)$. Suppose that for every $j \in J$,

$$\mathbb{P}_{ann}(E_j \,|\mathcal{S}''(x,y) \cap \mathcal{E}_n^c) \ge \log^{\gamma(K_8/3-1)} n, \tag{3.70}$$

where $K_8 := \log_{d-1}((1+\lambda_h)/2)$. Then, letting $Z \sim \mathrm{Bin}(\lfloor \log^{\gamma-3\kappa-18} n \rfloor, \log^{\gamma(K_8/3-1)} n)$, we have

$$\mathbb{P}_{ann}(\mathcal{E}_n^c \cap (\mathcal{S}''(x,y) \setminus \mathcal{S}^*(x,y))\,) \le \mathbb{P}(Z=0).$$

But if κ and γ/κ are large enough so that

$$\gamma - 3\kappa - 18 + \gamma(K_8/3 - 1) = \gamma K_8/3 - 3\kappa - 18 > 0,$$

we have $\lim_{n\to+\infty} \mathbb{P}(Z=0) = 0$ and this yields (3.69). Thus, we are left with showing (3.70). We split the proof of (3.70) in two parts. First, we prove that

$$\mathbb{P}_{ann}(v_j(0) \in \mathcal{C}_{z_j} \mid (\mathcal{S}''(x,y) \cap \mathcal{E}_n^c)\,) \ge \log^{(\gamma-2\kappa)(K_8/2-1)} n, \tag{3.71}$$

where $\mathcal{C}_{z_j}$ is the connected component of z_j in $E^{\ge h}_{\psi_{\mathcal{G}_n}} \cap B^*(z_j, a'_n - 2a_n)$. Second, we show that for some constant $K_9 > 0$ (uniquely depending on d and h),

$$\mathbb{P}_{ann}(\forall v \in P_j, \psi_{\mathcal{G}_n}(v) \ge h \mid (\mathcal{S}''(x,y) \cap \mathcal{E}_n^c \cap \{v_j(0) \in \mathcal{C}_{z_j}\})\,) \ge \log^{-K_9\kappa} n. \tag{3.72}$$

We prove that both hold for n large enough, uniformly in $v \in T_j$ and on realization of the $\psi_{\mathcal{G}_n}$ on $T_1 \cup \ldots \cup T_{j-1}$, as long as we are in $\mathcal{E}_n^c$ (so that we can apply Proposition 3.4.1).
(3.71) and (3.72) imply indeed that

$$\mathbb{P}_{ann}(E_j\,|\mathcal{S}''(x,y)\cap\mathcal{E}_n^c)\geq \log^{(\gamma-2\kappa)(K_8/2-1)-K_9\kappa} n \geq \log^{\gamma(K_8/3-1)} n$$

if γ/κ is large enough, which yields (3.70).

Part 1: proof of (3.71).

Since $|A_{j,k,i}|\leq n^{2/3}$ and $\mathrm{tx}(B_{\mathcal{G}_n}(A_{j,k,i},a_n))=\mathrm{tx}(A_{j,k,i})$ for all $k,i\geq 0$, we can apply Proposition 3.4.1 as below (3.54) to bound the difference between $\psi_{\mathcal{G}_n}$ on $\mathcal{C}_{z_j}$ and $\varphi_{\mathbb{T}_d}$ on an isomorphic subtree of $\mathbb{T}_d$, with the following coupling: $\varphi_{\mathbb{T}_d}(\circ):=\psi_{\mathcal{G}_n}(z_j)$, and then $\varphi_{\mathbb{T}_d}$ is defined as in Proposition 3.3.1 via $(\xi_{j,k,i})_{k,i\geq 0}$. Recall that on $\mathcal{S}''(x,y)$, we have that $\psi_{\mathcal{G}_n}(z_j)\geq h+(\log^{-1}n)/2$. By Proposition 3.3.8, for any $\delta>0$ and for large enough n,

$$\min_{a\geq h+(\log^{-1}n)/2}\mathbb{P}_a^{\mathbb{T}_d}\left(|\mathcal{Z}_{a_n'-2a_n}^{h+(\log^{-1}n)/2,+}|\geq(\lambda_h-\delta)^{a_n'-2a_n}\right)\geq\frac{p\mathbb{P}^{\mathbb{T}_d}(\mathcal{E}^+)}{2}$$

where $\mathbb{P}^{\mathbb{T}_d}(\mathcal{E}^+)>0$ (recall (3.23)) and

$$p:=\min\nolimits_{a\geq h+(\log^{-1}n)/2}\mathbb{P}_a^{\mathbb{T}_d}(\exists v\in B_{\mathbb{T}_d^+}(\circ,1),\,\varphi_{\mathbb{T}_d}(z)\geq h+1)>0.$$

Note in particular that for $\delta'>\frac{\log^{-1}n}{2}$ such that $\lambda_{h+\delta'}>\lambda_h-\delta$ (such δ' exists by Proposition 3.3.4, if n is large enough), $\mathcal{Z}_{a_n'-2a_n}^{h+\delta',+}\subseteq\mathcal{Z}_{a_n'-2a_n}^{h+(\log^{-1}n)/2,+}$.

Proposition 3.4.1 yields then

$$\mathbb{P}_{ann}\left(|\partial\mathcal{C}_{z_j}|\geq(\lambda_h-\delta)^{a_n'-2a_n}\,\middle|\mathcal{S}''(x,y)\cap\mathcal{E}_n^c\right)\geq\frac{p\mathbb{P}(\mathcal{E}^+)}{2}+o(1)\geq\frac{p\mathbb{P}(\mathcal{E}^+)}{3}.$$

By cylindrical symmetry of $B_{\mathbb{T}_d^+}(\circ,a_n'-2a_n)$, we even have

$$\begin{aligned}\mathbb{P}_{ann}\left(v(0)\in\partial\mathcal{C}_{z_j}\,\middle|\mathcal{S}''(x,y)\cap\mathcal{E}_n^c\right)\geq&\frac{p\mathbb{P}(\mathcal{E}^+)}{3}\frac{(\lambda_h-\delta)^{a_n'-2a_n}}{|\partial B_{\mathbb{T}_d^+}(\circ,a_n'-2a_n)|}\\ \geq&\frac{p\mathbb{P}(\mathcal{E}^+)}{3}\left(\frac{\lambda_h-\delta}{d-1}\right)^{a_n'-2a_n}.\end{aligned}$$

Since $K_8=\log_{d-1}((1+\lambda_h)/2)$, taking δ small enough yields (3.71).

Part 2: proof of (3.72).

Denote $v_j(1),\ldots,v_j(a_n-1)$ the vertices from $v_j(0)$ to $v_j(a_n)$ on the path P_j. Remark that it suffices to prove that there exists a constant $K_9>0$ such that for n large enough, for every $k\in\{1,\ldots,a_n\}$,

$$\mathbb{P}_{ann}\left(\psi_{\mathcal{G}_n}(v_j(k))\geq h|\left(\mathcal{S}''(x,y)\cap\mathcal{E}_n^c\cap\{\psi_{\mathcal{G}_n}(v_j(k-1))\geq h\}\right)\right)\geq(d-1)^{-K_9}. \tag{3.73}$$

In the notation of (3.68), $v_j(k)=y_{j,k+a_n'-2a_n,1}$ for $1\leq k\leq a_n$. Write $A_k:=A_{j,k+a_n'-2a_n,1}$. Suppose that for n large enough and all $k\in\{1,\ldots,a_n\}$, on $\mathcal{S}''(x,y)\cap\mathcal{E}_n^c\cap\{\psi_{\mathcal{G}_n}(v_j(k-1))\geq h\}$:

$$\mathbb{E}^{\mathcal{G}_n}[\psi_{\mathcal{G}_n}(v_j(k))|\sigma(A_k)]>-|h|-1 \tag{3.74}$$

and

$$\mathrm{Var}^{\mathcal{G}_n}(\psi_{\mathcal{G}_n}(v_j(k))|\sigma(A_k))>\frac{1}{d-1}. \tag{3.75}$$

Then (3.73) holds with

$$K_9 := -\log_{d-1} \mathbb{P}(Y \geq (|h| + \tfrac{|h|+1}{d-1})/\sqrt{d-1}\,),$$

where $Y \sim \mathcal{N}(0,1)$. Thus, it is enough to establish (3.74) and (3.75).
For $k \geq 1$, note that by construction of $B^*(z_j, a'_n)$, $v_j(k-1)$ and $v_j(a_n)$ are the only vertices of ∂A_k at distance less than a_n of $v_j(k)$. Let $(X_s)_{s\geq 0}$ be a discrete time SRW started at $v_j(k)$, and $\tau := \inf\{s \geq 0,\, d_{\mathcal{G}_n}(v_j(k), X_s) \geq a_n\}$. Write H for the hitting time of A_k by (X_s). Letting

$$a_1 := \mathbf{P}^{\mathcal{G}_n}_{v_j(k)}(X_H = v_j(k-1), H < \tau) \text{ and } a_2 := \mathbf{P}^{\mathcal{G}_n}_{v_j(k)}(X_H = v_j(a_n), H < \tau),$$

we get as in the proof of Proposition 3.4.1 that for large enough n, for every realization of $\mathcal{G}_n$ and $\psi_{\mathcal{G}_n}(A_k)$ in $\mathcal{S}''(x,y) \cap \mathcal{E}_n^c \cap \{\psi_{\mathcal{G}_n}(v_j(k-1)) \geq h\}$:

$$\mathbb{E}^{\mathcal{G}_n}[\psi_{\mathcal{G}_n}(v_j(k))|\sigma(A_k)] > a_1\psi_{\mathcal{G}_n}(v_j(k-1)) + a_2\psi_{\mathcal{G}_n}(v_j(a_n)) - \log^{-1} n.$$

Since $0 \leq a_1 + a_2 \leq 1$ and $\min(\psi_{\mathcal{G}_n}(v_j(k-1)), \psi_{\mathcal{G}_n}(v_j(a_n))) \geq h \geq -|h|$, (3.74) follows.
Split $V := \mathrm{Var}^{\mathcal{G}_n}(\psi_{\mathcal{G}_n}(v_j(k))|\sigma(A_k))$ as follows:

$$\begin{aligned} V = & G_{\mathcal{G}_n}(v_j(k), v_j(k)) - \mathbf{E}^{\mathcal{G}_n}_{v_j(k)}\left[G_{\mathcal{G}_n}(v_j(k), X_H)\,\mathbf{1}_{\{H<\tau\}}\right] - \mathbf{E}^{\mathcal{G}_n}_{v_j(k)}\left[G_{\mathcal{G}_n}(v_j(k), X_H)\,\mathbf{1}_{\{H\geq\tau\}}\right] \\ & + \frac{\mathbf{E}^{\mathcal{G}_n}_{v_j(k)}[H]}{\mathbf{E}^{\mathcal{G}_n}_{\pi_n}[H]}\mathbf{E}^{\mathcal{G}_n}_{\pi_n}\left[G_{\mathcal{G}_n}(v_j(k), X_H)\right]. \end{aligned}$$

By (3.11), (3.8) and (3.9), if κ is large enough, for n large enough,

$$\begin{aligned} G_{\mathcal{G}_n}(v_j(k), v_j(k)) - a_1 G_{\mathcal{G}_n}(v_j(k), v_j(k-1)) - a_2 G_{\mathcal{G}_n}(v_j(k), v_j(a_n)) &> \frac{d-1}{d-2} - \frac{a_1+a_2}{d-2} - \log^{-1} n \\ &\geq \frac{1}{d-2} - \log^{-1} n. \end{aligned}$$

As below (3.45), we get that

$$\left|\mathbf{E}^{\mathcal{G}_n}_{v_j(k)}\left[G_{\mathcal{G}_n}(v_j(k), X_H)\,\mathbf{1}_{\{H<\tau\}}\right] - a_1 G_{\mathcal{G}_n}(v_j(k), v_j(k-1)) - a_2 G_{\mathcal{G}_n}(v_j(k), v_j(a_n))\right| \leq \log^{-1} n$$

and that

$$\left|\mathbf{E}^{\mathcal{G}_n}_{v_j(k)}\left[G_{\mathcal{G}_n}(v_j(k), X_H)\,\mathbf{1}_{\{H\geq\tau\}}\right] - \frac{\mathbf{E}^{\mathcal{G}_n}_{v_j(k)}[H]}{\mathbf{E}^{\mathcal{G}_n}_{\pi_n}[H]}\mathbf{E}^{\mathcal{G}_n}_{\pi_n}\left[G_{\mathcal{G}_n}(v_j(k), X_H)\right]\right| \leq \log^{-1} n.$$

These three inequalities imply that for large enough n, for every realization of $\mathcal{G}_n$ and $\psi_{\mathcal{G}_n}(A_k)$ in $\mathcal{S}''(x,y) \cap \mathcal{E}_n^c \cap \{\psi_{\mathcal{G}_n}(v_j(k-1)) \geq h\}$:

$$\mathrm{Var}^{\mathcal{G}_n}(\psi_{\mathcal{G}_n}(v_j(k))|\sigma(A_k)) \geq \tfrac{1}{d-2} - 3\log^{-1} n > \tfrac{1}{d-1}.$$

This shows (3.75) and the proof is complete. □

3.6.2 Average number of connections in $E^{\geq h}_{\psi_{\mathcal{G}_n}}$

Write $x \overset{h}{\leftrightarrow} y$ if x and y are in the same connected component of $E^{\geq h}_{\psi_{\mathcal{G}_n}}$, for $x, y \in V_n$. In this section, we prove (3.2) of Theorem 3.1.1 via an argument on the number of pairs of vertices such that $x \overset{h}{\leftrightarrow} y$. Let S_n be the set of pairs of distinct $x, y \in V_n$ such that $x \overset{h}{\leftrightarrow} y$. Let $A_n \subseteq V_n$ be the set of vertices x such that $|\mathcal{C}_x^{\mathcal{G}_n,h}| \leq n^{1/2}$.

We first suppose that the following two Lemmas hold, and show (3.2). Then, we derive them from Propositions 3.5.4 and 3.6.2, using a second moment argument.

Lemma 3.6.3. *For every $\varepsilon > 0$, $\lim_{n\to+\infty} \mathbb{P}_{ann}\left(|A_n| \geq (1-\eta(h)-\varepsilon)n\right) = 1$.*

Lemma 3.6.4. *For every $\varepsilon > 0$, $\lim_{n\to+\infty} \mathbb{P}_{ann}(|S_n| \geq (\eta(h)^2/2-\varepsilon)n^2) = 1$.*

Proof of (3.2). Fix $\varepsilon > 0$. Remark that every connected component of $E^{\geq h}_{\psi_{\mathcal{G}_n}}$ is either included in A_n, or does not intersect A_n. Let $(\widehat{\gamma}_i^{(n)})_{i\geq 1}$ (resp. $(\gamma_i^{(n)})_{i\geq 1}$) be the sizes of the connected components in A_n (resp. not in A_n), listed in decreasing order of size.

Let $\mathcal{E}_n := \{|A_n| \geq (1-\eta(h)-\varepsilon)n\} \cap \{|S_n| \geq (\eta(h)^2/2-\varepsilon)n^2\}$. On $\mathcal{E}_n$, we have that

$$\textstyle\sum_{i\geq 1} \widehat{\gamma}_i^{(n)}(\widehat{\gamma}_i^{(n)}-1) + \sum_{i\geq 1} \gamma_i^{(n)}(\gamma_i^{(n)}-1) = 2S_n \geq \eta(h)^2n^2 - 2\varepsilon n^2.$$

Moreover, we have by definition of A_n:

$$\textstyle\sum_{i\geq 1} \widehat{\gamma}_i^{(n)}(\widehat{\gamma}_i^{(n)}-1) \leq \sum_{i\geq 1} \widehat{\gamma}_i^{(n)}\sqrt{n} \leq n^{7/4}.$$

Thus, for n large enough,

$$\gamma_1^{(n)}(|V_n| - |A_n|) \geq \sum_{i\geq 1} \gamma_i^{(n)}(\gamma_i^{(n)}-1) \geq (\eta(h)^2 - 3\varepsilon)n^2.$$

But $|V_n| - |A_n| \leq (\eta(h)+\varepsilon)n$, so that

$$\gamma_1^{(n)} \geq \frac{\eta(h)^2 - 3\varepsilon}{\eta(h)+\varepsilon} n \geq ((\eta(h) - 4\eta(h)^{-1}\varepsilon)n.$$

Since $\gamma_1^{(n)}$ is the cardinality of a set included in $V_n \setminus A_n$, one has $\gamma_1^{(n)} \leq (\eta(h)+\varepsilon)n$. Note that for n large enough, $\gamma_1^{(n)} > \sqrt{n} \geq \widehat{\gamma}_1^{(n)}$. Therefore, $\gamma_1^{(n)} = |\mathcal{C}_1^{(n)}|$, and we have

$$((\eta(h) - 4\eta(h)^{-1}\varepsilon)n \leq |\mathcal{C}_1^{(n)}| \leq (\eta(h)+\varepsilon)n$$

on $\mathcal{E}_n$. By Lemmas 3.6.3 and 3.6.4, $\lim_{n\to+\infty} \mathbb{P}_{ann}(\mathcal{E}_n) = 1$. Since ε was arbitrary, the proof is complete. ☐

Remark 3.6.5. *Note that we have $|\mathcal{C}_2^{(n)}| = \max(\gamma_2^{(n)}, \widehat{\gamma}_1^{(n)})$. Since on $\mathcal{E}_n$, $\widehat{\gamma}_1^{(n)} \leq \sqrt{n}$ and*

$$\gamma_2^{(n)} \leq |V_n| - |A_n| - \gamma_1^{(n)} \leq (1 + 4\eta(h)^{-1})\varepsilon n,$$

we get that $|\mathcal{C}_2^{(n)}|/n \overset{\mathbb{P}_{ann}}{\longrightarrow} 0$.

Proof of Lemma 3.6.3. Let A'_n be the set of vertices such that their lower exploration (Section 3.5.2) is aborted. By Remark 3.5.2, for n large enough, $A'_n \subseteq A_n$, and it is enough to prove the Lemma for A'_n instead of A_n.
Let $\varepsilon \in (0,1)$. By Proposition 3.5.4, for n large enough and every $x \in V_n$, we have

$$|\mathbb{P}_{ann}(x \in A'_n) - (1-\eta(h))| \leq \varepsilon. \tag{3.76}$$

We claim that for n large enough, for all distinct $x, y \in V_n$,

$$|\mathrm{Cov}_{ann}(\mathbf{1}_{x\in A'_n}, \mathbf{1}_{y\in A'_n})| \leq 4\varepsilon. \tag{3.77}$$

Indeed, $\mathrm{Cov}_{ann}(\mathbf{1}_{x\in A'_n}, \mathbf{1}_{y\in A'_n}) = \mathbb{P}_{ann}(x, y \in A'_n) - \mathbb{P}_{ann}(x \in A'_n)\mathbb{P}_{ann}(y \in A'_n)$. Then by (3.76), we have

$$|\mathbb{P}_{ann}(x \in A'_n)\mathbb{P}_{ann}(y \in A'_n) - (1-\eta(h))^2| \leq 2\varepsilon + \varepsilon^2 \leq 3\varepsilon.$$

Perform successively the lower explorations from x and then from y as in Section 3.5.2 (with the additional condition C5). We get $\mathbb{P}_{ann}(\text{C5 happens}) = o(1)$ in the same way than (3.53). Then, revealing $\psi_{\mathcal{G}_n}$ on $R_x \cup R_y$ and comparing it to $\varphi_{\mathbb{T}_d}$ as below (3.54), we obtain

$$|\mathbb{P}_{ann}(x, y \in A'_n) - (1-\eta(h))^2| \leq \varepsilon.$$

This shows (3.77).
We now apply Bienaymé-Chebyshev's inequality:

$$\begin{aligned}
\mathbb{P}_{ann}(|A'_n| \leq (1-\eta(h) - 2\varepsilon^{1/4})n) &\leq \mathbb{P}_{ann}(|\,|A'_n| - \mathbb{E}_{ann}[|A'_n|]\,| \geq \varepsilon^{1/4}n) \\
&\leq \frac{1}{\sqrt{\varepsilon}n^2}\left(\sum_{x,y\in V_n} \mathrm{Cov}_{ann}(\mathbf{1}_{x\in A'_n}, \mathbf{1}_{y\in A'_n})\right) \\
&\leq \frac{n + n(n-1)4\varepsilon}{\sqrt{\varepsilon}n^2} \\
&\leq 5\sqrt{\varepsilon}
\end{aligned}$$

for n large enough. Since ε can be taken arbitrarily small, the proof is complete. □

Proof of Lemma 3.6.4. Let $\varepsilon \in (0,1)$. Denote S^*_n the set of pairs $x, y \in V_n$ such that $\mathcal{S}^*(x,y)$ holds. Since $S^*_n \subseteq S_n$, it is enough to prove the Lemma for S^*_n instead of S_n. First, by Proposition 3.6.2,

$$\mathbb{E}_{ann}[|S^*_n|] \geq (\eta(h)^2 - \varepsilon)\tfrac{n(n-1)}{2} \geq (\eta(h)^2/2 - \varepsilon)n^2$$

for large enough n. Second, we claim that for n large enough and all distinct $x, y, w, t \in V_n$,

$$|\mathrm{Cov}_{ann}(\mathbf{1}_{\mathcal{S}^*(x,y)}, \mathbf{1}_{\mathcal{S}^*(w,t)})| \leq 2\varepsilon. \tag{3.78}$$

Remark that if this holds, then we can conclude by a second moment computation as in the proof of Lemma 3.6.3. We have

$$\mathrm{Cov}_{ann}(\mathbf{1}_{\mathcal{S}^*(x,y)}, \mathbf{1}_{\mathcal{S}^*(w,t)}) = \mathbb{P}_{ann}(\mathcal{S}^*(x,y) \cap \mathcal{S}^*(w,t)) - \mathbb{P}_{ann}(\mathcal{S}^*(x,y))\mathbb{P}_{ann}(\mathcal{S}^*(w,t)).$$

By Proposition 3.6.2, for n large enough,

$$|\mathbb{P}_{ann}(\mathcal{S}^*(x,y))\mathbb{P}_{ann}(\mathcal{S}^*(w,t)) - \eta(h)^4| \leq \varepsilon. \tag{3.79}$$

Now, perform successively the exploration of Section 3.5.1 from x, then from y, then from z and finally from t (via an array of i.i.d. standard normal variables $(\xi_{u,v})_{u\in\{x,y,w,t\},v\in\mathbb{T}_d}$). We add the following condition: for any $u \in \{x, y, w, t\}$, the exploration from v is stopped as soon as it meets a vertex seen in a previous exploration. The probability that this happens is $o(1)$ by Remark 3.5.2 and (3.14), since $o(\sqrt{n})$ vertices and half-edges are revealed during these four explorations. Therefore, as for (3.54), we get that for n large enough,

$$\mathbb{P}_{ann}(\text{the explorations from } x, y, z, t \text{ are all successful}) \in (\eta(h)^4 - \varepsilon/2, \eta(h)^4 + \varepsilon/2). \tag{3.80}$$

If these explorations are successful, develop balls from ∂T_x to ∂T_y as described in Section 3.6.1, with $Q_j := R_x \cup R_y \cup R_w \cup R_t \cup B_j^*$ for $z_j \in \partial T_x$. Then do the same from ∂T_w to ∂T_t, this time with $Q_j := R_x \cup R_y \cup (\cup_{z\in\partial T_x} B^*(z, a_n')) \cup R_w \cup R_t \cup B_j^*$ for $z_j \in \partial T_w$. Finally, reveal $\psi_{\mathcal{G}_n}$ on T_x, T_y, T_w, T_t and on the joining balls from T_x to T_y and from T_w to T_t, in that order.
One can adapt readily the proof of Lemma 3.6.1 to show that with $\mathbb{P}_{ann}$-probability $1-o(1)$, if the four explorations are successful then there are at least $\log^{\gamma-3\kappa-18} n$ joining balls from ∂T_x (resp. ∂T_w) to ∂T_y (resp. ∂T_t). Note in particular that the estimations of (3.61), (3.63) and (3.64) still hold. It is also straightforward to carry the arguments of the proof of Proposition 3.6.2, and we finally have

$$|\mathbb{P}_{ann}(\mathcal{S}^*(x,y) \cap \mathcal{S}^*(w,t)) - \mathbb{P}_{ann}(\text{the explorations from } x, y, w, t \text{ are all successful})| \leq \varepsilon/2.$$

Together with (3.79) and (3.80), this yields (3.78). □

3.7 Uniqueness of the giant component

In this Section, we prove (3.3). We start by the lower bound in Section 3.7.1, showing the existence $\mathbb{P}_{ann}$-w.h.p. of a component (different from $\mathcal{C}_1^{(n)}$) having $\Theta(\log n)$ vertices.
Then, to show that $|\mathcal{C}_2^{(n)}| = O(\log n)$ $\mathbb{P}_{ann}$-w.h.p., we perform an exploration of a new kind, starting from some $x \in V_n$. It consists of three phases (Sections 3.7.2 to 3.7.4), during which we assign a **pseudo-GFF** $\widehat{\psi_{\mathcal{G}_n}}$ to the vertices that we visit. $\widehat{\psi_{\mathcal{G}_n}}$ is defined via a recursive construction that mimics Proposition 3.3.1, as long as there are no cycles (in Section 3.5, the analogous of the pseudo-GFF was $\varphi_{\mathbb{T}_d}$ on $\mathfrak{T}_x$). Finally, in Section 3.7.5, we reveal the true values of $\psi_{\mathcal{G}_n}$ one by one on the set of vertices we have explored via Proposition 3.2.4, and show that either $|\mathcal{C}_x^{\mathcal{G}_n,h}| = O(\log n)$, or $|\mathcal{C}_x^{\mathcal{G}_n,h}| = \Theta(n)$, in which case $\mathcal{C}_x^{\mathcal{G}_n,h} = \mathcal{C}_1^{(n)}$ by Remark 3.6.5. Contrary to Section 3.5, we need this alternative to hold for *every* $x \in V_n$, $\mathbb{P}_{ann}$-w.h.p. By a

union bound, it is enough to prove that

$$\text{for } x \in V_n,\ \mathbb{P}_{ann}(\{|\mathcal{C}_x^{\mathcal{G}_n,h}| = O(\log n)\} \cup \{|\mathcal{C}_x^{\mathcal{G}_n,h}| = \Theta(n)\}) = o(1/n). \tag{3.81}$$

Let us sketch this exploration in the lines below.

First phase (Section 3.7.2). We explore the connected component $\mathcal{C}$ of $x \in V_n$ in the set $\{y \in V_n, \widehat{\psi_{\mathcal{G}_n}}(y) \geq h - n^{-a}\}$ for some constant $a > 0$. More precisely, we give a mark to each vertex y such that $|\widehat{\psi_{\mathcal{G}_n}}(y) - h| \leq n^{-a}$, and explore each connected component $\mathcal{C}'$ of $\mathcal{C} \setminus \mathcal{M}$, where $\mathcal{M}$ is the set of marked vertices, until

- (i) $\mathcal{C}'$ is fully explored and has no more than $O(\log n)$ vertices, or

- (ii) $\lfloor K \log n \rfloor$ vertices of $\mathcal{C}'$ have been seen but not yet explored, for some constant K fixed in the second phase.

We replace a_n of (3.33) by a "security radius" $r_n = \Theta(\log n)$. Adapting Proposition 3.4.1 (Lemma 3.7.7), this allows us in Section 3.7.5 to bound the difference between $\widehat{\psi_{\mathcal{G}_n}}$ and $\psi_{\mathcal{G}_n}$ by n^{-a}, so that for every connected component $\mathcal{C}'$ of $\mathcal{C} \setminus \mathcal{M}$, either $\mathcal{C}' \subseteq \mathcal{C}_x^{\mathcal{G}_n,h}$ or $\mathcal{C}' \cap \mathcal{C}_x^{\mathcal{G}_n,h} = \emptyset$.
If we kept a_n, $\widehat{\psi_{\mathcal{G}_n}}$ would approximate $\psi_{\mathcal{G}_n}$ only with precision $\log^{-\Theta(1)} n$. With probability $\Theta(1/n)$, there would be too much vertices y such that $|\widehat{\psi_{\mathcal{G}_n}}(y) - h| \leq \log^{-\Theta(1)} n$, hence for which we cannot know by anticipation whether they will be in $\mathcal{C}_x^{\mathcal{G}_n,h}$ or not.
Moreover, we do not have $\mathbb{P}(\text{C1 happens}) = O(1/n)$ as soon as the number of vertices explored goes to infinity with n. We will need to accept the possible occurrence of one cycle. When this happens, we have to define $\widehat{\psi_{\mathcal{G}_n}}$ in a slightly different manner. In Section 3.7.5, we need a variant of Lemma 3.7.7 to control the difference between $\widehat{\psi_{\mathcal{G}_n}}$ and $\psi_{\mathcal{G}_n}$ (Lemma 3.7.8).

Second phase (Section 3.7.3). If (i) happens for every component $\mathcal{C}'$, the exploration is over. For each $\mathcal{C}'$ such that (ii) happens, we explore its $\lfloor K \log n \rfloor$ remaining vertices, this time in a fashion similar to Section 3.5.1. Each of these explorations has a probability bounded away from 0 to be successful. If K is large enough, with probability at least $1 - o(1/n)$, at least one of these explorations is successful, and has a boundary of size $\Theta(n^{1/2} b_n)$.

Third phase(Section 3.7.4). For every $\mathcal{C}'$ such that (ii) happens, we show that the successful exploration of the second phase is connected to a positive proportion of the vertices of V_n, via an adaptation of the joint exploration in Section 3.6.1. This yields (3.81).

3.7.1 Lower bound

In this section, we prove the existence of $K_0 > 0$ such that

$$\mathbb{P}_{ann}(|\mathcal{C}_2^{(n)}| \geq K_0^{-1} \log n) \to 1. \tag{3.82}$$

To show the existence of $x \in V_n$ such that the size of its connected component is exactly of order $\log n$ requires an exploration in which we compare $\mathcal{C}_x^{\mathcal{G}_n,h}$ to both $\mathcal{C}_\circ^{h+\log^{-1} n}$ and $\mathcal{C}_\circ^{h-\log^{-1} n}$. Then, the proof strategy simply consists in performing explorations from different vertices of V_n, one after another, until one of them is successful. Proposition 3.3.6 hints that the probability that $\mathcal{C}_\circ^h$ has size $c\log n$ should be of order $n^{-f(c)}$, where f is an unknown function of $c>0$, with $f(c)\underset{c\to 0}{\longrightarrow} 0$. Hence, for $K_0^{-1}=c$ small enough, performing at most $n^{1/10}$ explorations will be enough.

The exploration. Let $K>0$. For $x\in V_n$, we modify the lower exploration of Section 3.5.2, replacing C2 by $|B_{\mathcal{C}_\circ^{h-\log^{-1}n}}(\circ,k-1)|\geq 2K\log n$. If at some step k, the lower exploration is stopped exclusively because of C3, and if $|B_{\mathcal{C}_\circ^{h+\log^{-1}n}}(\circ,k-1)|\geq K\log n$, say that it is successful.

Pick $x_0\in V_n$. Denote S_0 the set of vertices seen during its lower exploration (i.e. vertices having at least one half-edge paired during the exploration). For $0\leq i\leq \lfloor n^{1/10}\rfloor$, if the lower exploration from x_i is not successful, let S_i be the set of vertices seen in the lower explorations of $x_0,\dots,x_i$. Pick $x_{i+1}\in V_n\setminus S_i$ and perform its lower exploration, stopping it if a vertex of S_i is seen, in which case it is not successful. Let $\mathcal{E}_{i,n}:=\{$the lower exploration from x_i is successful$\}$ and $\mathcal{E}_n:=\cup_{i=0}^{\lfloor n^{1/10}\rfloor}\mathcal{E}_{i,n}$.

Suppose that the following result holds:

Lemma 3.7.1. *If K is small enough, then* $\lim_{n\to+\infty}\mathbb{P}_{ann}(\mathcal{E}_n)=1$.

On $\mathcal{E}_n$, let $i_0\geq 1$ be such that the lower exploration from x_{i_0} is successful. Applying Proposition 3.4.1 as below (3.54), we get that

$$\mathbb{P}_{ann}(\mathcal{C}_\circ^{h+\log^{-1}n}\subseteq \Phi(\mathcal{C}_{x_{i_0}}^{\mathcal{G}_n,h})\subseteq \mathcal{C}_\circ^{h-\log^{-1}n}),$$

where Φ is an isomorphism from $\mathcal{C}_{x_{i_0}}^{\mathcal{G}_n,h}$ to $\mathbb{T}_d$. In this case, $K\log n\leq |\mathcal{C}_{x_{i_0}}^{\mathcal{G}_n}|\leq 2K\log n$. With high $\mathbb{P}_{ann}$-probability, $|\mathcal{C}_1^{(n)}|>2K\log n$ by (3.2), so that $\mathcal{C}_{x_{i_0}}^{\mathcal{G}_n,h}\neq \mathcal{C}_1^{(n)}$. This yields (3.82) with $K=K_0^{-1}$. It remains to establish the Lemma.

Proof of Lemma 3.7.1. Clearly, it is enough to show that for K small enough, for n large enough and every $1\leq i\leq n^{1/10}$,

$$\mathbb{P}_{ann}(\mathcal{E}_{i,n}|\cap_{j=0}^{i-1}\mathcal{E}_{j,n}^c)\geq n^{-1/11}, \tag{3.83}$$

since it would imply that $\mathbb{P}_{ann}(\mathcal{E}_n)\geq 1-(1-n^{-1/11})^{\lfloor n^{1/10}\rfloor}\underset{n\to+\infty}{\longrightarrow} 1$.

For $n\geq 1$ and $1\leq i\leq\lfloor n^{1/10}\rfloor$, we have

$$\mathcal{E}_{i,n}\supseteq \mathcal{E}_{\text{line},n}\setminus(\{\text{C1 happens}\}\cup\mathcal{E}_{i,n,\text{meet}}),$$

where

$\mathcal{E}_{i,n,\text{meet}}$:=$\{$the lower exploration from x_i meets $S_{i-1}\}$,$\mathcal{E}_{\text{line},n}$:=$\{\mathcal{C}_\circ^{h+\log^{-1}n}$=$\mathcal{C}_\circ^{h-\log^{-1}n}$=$L_{\lfloor K\log n\rfloor}\}$ and for any $k\geq 1$, L_k is a "line" subtree of $\mathbb{T}_d$, i.e. it is rooted at $\circ$, has $k+1$ vertices and total height k.

During each exploration, less than $n^{1/10}$ vertices and half-edges are seen for n large enough. Thus $|S_i|\leq n^{1/5}$. Hence, for $i\geq 1$, by (3.14) with $k=1$, $m\leq n^{1/10}$, $m_0=1$ and $m_E,m_1\leq n^{1/5}$, for n large enough:

$$\mathbb{P}_{ann}(\mathcal{E}_{i,n,\text{meet}}) + \mathbb{P}_{ann}(\text{C1 happens}) \leq n^{-1/2}.$$

Thus to establish (3.83), it only remains to show that for K small enough and n large enough,

$$\mathbb{P}_{ann}(\mathcal{E}_{\text{line},n}) \geq n^{-1/12}.$$

Let $v_1, \ldots, v_d$ be the children of $\circ$ in $\mathbb{T}_d$. Remark that for n large enough,

$$\mathbb{P}_{ann}(\mathcal{E}_{\text{line},n}) \geq \mathbb{P}^{\mathbb{T}_d}(\varphi_{\mathbb{T}_d}(\circ) \in [h+1/2, h+1]) p p'^{\lfloor K \log n \rfloor} p'',$$

where

$$p := \inf_{a \in [h+1/2, h+1]} \mathbb{P}_a^{\mathbb{T}_d}(\{\varphi_{\mathbb{T}_d}(v_1) \in [h+1/2, h+1]\} \cap \{\forall i \in \{2, \ldots, d\}, \varphi_{\mathbb{T}_d}(v_i) < h-1\}),$$

$$p' := \inf_{a \in [h+1/2, h+1]} \mathbb{P}_a^{\mathbb{T}_d}(\{\varphi_{\mathbb{T}_d}(v_1) \in [h+1/2, h+1]\} \cap \{\forall i \in \{2, \ldots, d-1\}, \varphi_{\mathbb{T}_d}(v_i) < h-1\}),$$

$$p'' := \inf_{a \in [h+1/2, h+1]} \mathbb{P}_a^{\mathbb{T}_d}(\forall i \in \{1, \ldots, d-1\}, \varphi_{\mathbb{T}_d}(v_i) < h-1).$$

Using Proposition 3.3.1, one checks easily that $p, p', p'' > 0$. Taking $K < -(12 \log p')^{-1}$ yields the result. □

3.7.2 First phase

In this section, we define the first phase of the exploration, and show that it is successful with $\mathbb{P}_{ann}$-probability $1 - n^{-5/4}$ (Proposition 3.7.2). We will need a variant of Lemma 3.3.9, namely Lemma 3.7.3. We postpone its statement and proof to the end of this section.
Let $a, K, K' > 0$. For every $n \in \mathbb{N}$, define

$$r_n := \lfloor 0.05 \log_{d-1} n \rfloor \tag{3.84}$$

Let $\delta \in (0, h_\star - h)$ and $\ell \in \mathbb{N}$ be such that the conclusion of Lemma 3.7.3 holds.

The exploration. Let $x \in V_n$.
I - We first assume that we do not meet any cycle throughout the exploration. Let $\mathcal{M}$ be the set of marked vertices. Initially, $\mathcal{M} = \emptyset$. Let $\widehat{\psi_{\mathcal{G}_n}}(x) \sim \mathcal{N}(0, \frac{d-1}{d-2})$. If $\varphi_{\mathbb{T}_d}(\circ) < h - n^{-a}$, stop the exploration. Else, give a mark to x (hence add it to $\mathcal{M}$).
While $\mathcal{M} \neq \emptyset$, pick $y \in \mathcal{M}$ in an arbitrary way and proceed to its **subexploration**.

The subexploration of y. Let T_y be the subexploration tree, that we will build by adding subtrees of depth ℓ in a breadth-first way. Initially $T_y = \{y\}$.

While $1 \leq |\partial T_y| \leq K \log n$, perform a **step**: take $y_1 \in \partial T_y$ of minimal height and if $y_1 \neq x$, let $\overline{y_1}$ be its only neighbour where $\widehat{\psi_{\mathcal{G}_n}}$ has already been defined. Note that if $y_1 \neq y$, $\overline{y_1}$ is the parent of y_1 in T_y. Reveal all the edges of $B_{\mathcal{G}_n}(y_1, \overline{y_1}, r_n + \ell)$, where we recall that $B_{\mathcal{G}_n}(y_1, \overline{y_1}, r_n + \ell)$ is the graph obtained by taking all paths of length $r_n + \ell$ starting at y_1 and not going through $\overline{y_1}$.

Since we suppose that no cycle arises, $B_{\mathcal{G}_n}(y_1, \overline{y_1}, \ell)$ is a tree, that we root at y_1. If $y_1 = y = x$, replace $B_{\mathcal{G}_n}(y_1, \overline{y_1}, r_n + \ell)$ and $B_{\mathcal{G}_n}(y_1, \overline{y_1}, \ell)$ by $B_{\mathcal{G}_n}(x, r_n + \ell)$ and $B_{\mathcal{G}_n}(x, \ell)$ respectively.
We construct $T_y(y_1)$, the subtree of $B_{\mathcal{G}_n}(y_1, \overline{y_1}, \ell)$ in $\{z, \widehat{\psi_{\mathcal{G}_n}}(z) \geq h + n^{-a}\}$. We start with $T_y(y_1) = \{y_1\}$.
For $k = 1, 2, ...\ell - 1$ successively, denote $y_{k,1}, \ldots, y_{k,m}$ the children of the $(k-1)$-th generation of $T_y(y_1)$. Let $(\xi_{y_1,k,i})_{k,i \geq 0}$ be an array of i.i.d. variables of law $\mathcal{N}(0,1)$, independent of everything else. Set

$$\widehat{\psi_{\mathcal{G}_n}}(y_{k,i}) := \frac{1}{d-1}\widehat{\psi_{\mathcal{G}_n}}(\overline{y_{k,i}}) + \sqrt{\frac{d}{d-1}}\xi_{y_1,k,i}. \tag{3.85}$$

Add $y_{k,i}$ to $T_y(y_1)$ if $\widehat{\psi_{\mathcal{G}_n}}(y_{k,i}) \geq h + n^{-a}$, and give a mark to $y_{k,i}$ (and thus add it to $\mathcal{M}$) if $h - n^{-a} \leq \widehat{\psi_{\mathcal{G}_n}}(y_{k,i}) < h + n^{-a}$.
Finally, include $T_y(y_1)$ in T_y, add the vertices of $\partial T_y(y_1)$ to N_y and take y_1 away from ∂T_y. The step is then over.

If $|\partial T_y| \notin [1, K \log n]$, the subexploration is finished. Say that it is **fertile** if $|\partial T_y| > K \log n$, and **infertile** else (hence if $\partial T_y = \emptyset$).

If $\mathcal{M} = \emptyset$, the exploration from y is finished.

II - Suppose now that a unique cycle C arises in the subexploration of y, when revealing the pairings of $B_{\mathcal{G}_n}(y_1, \overline{y_1}, r_n + \ell)$ in the step from y_1, for some $y, y_1 \in V_n$. Let $m := |C|$ be the number of vertices in C. There are two cases.
Case 1: When C is discovered, there are already k consecutive vertices $y_1, \ldots, y_k$ of C where $\widehat{\psi_{\mathcal{G}_n}}$ has been defined, for some $1 \leq k \leq m-1$. Reveal $B_{\mathcal{G}_n}(C, r_n)$. Denote $z_1, \ldots, z_{m-k}$ the remaining vertices of C, such that $z_1 \neq y_2$ is a neighbour of y_1, and z_i is a neighbour of z_{i-1} for $i \geq 2$. Give a mark to $z_1, \ldots, z_{m-k}$ and y_1. Take y_1 away from ∂T_y. If y_k was in $\partial T_{\widetilde{y}}$ for some $\widetilde{y}$ whose subexploration was performed previously, take it away from that set.

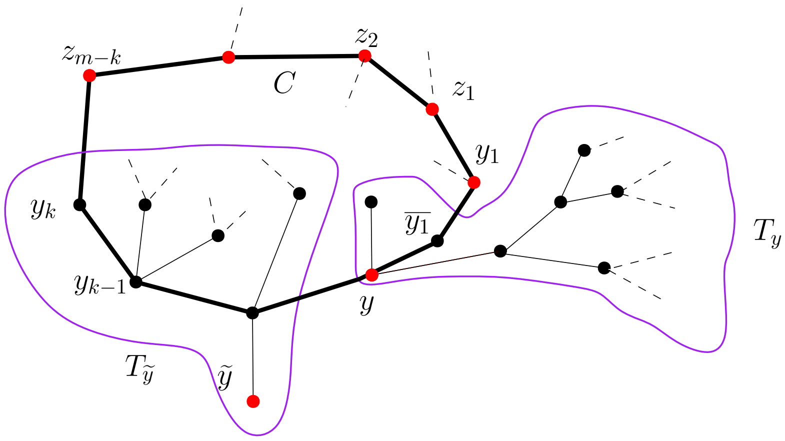

Figure 4. Case 1. Marked vertices are in red. C consists of the thick edges. T_y and $T_{\widetilde{y}}$ are delimited by the purple contours. Remark that we could have $y = \widetilde{y}$ (it is not the case here).

We now define $\widehat{\psi_{\mathcal{G}_n}}$ on the z_i's. To do so, we mimic a recursive construction of the GFF on G_m, the infinite connected d-regular graph G_m having a unique cycle C_m of length m (such a construction always exists on a transient graph by Proposition 3.3.2, Proposition 3.3.1 being a particular case of that on $\mathbb{T}_d$). G_m consists of a cycle C_m of length m, with $d-2$ copies of $\mathbb{T}_d^+$ attached to each vertex of C_m, thus it is clear that the SRW is transient and that the Green function and the GFF are well-defined.
Let $u_k, \dots, u_1, v_1, \dots, v_{m-k}$ be the vertices of C_m, listed consecutively. Let $U := G_m \setminus \{u_1, \dots, u_k\}$, $(X_j)_{j\geq 0}$ a SRW on G_m and recall that T_U is the exit time of U. Define

$$\alpha := \mathbf{P}_{v_1}^{G_m}(X_{T_U} = u_1, T_U < +\infty),\ \beta := \mathbf{P}_{v_1}^{G_m}(X_{T_U} = u_k, T_U < +\infty)\mathbf{1}_{\{k>1\}}$$

$$\text{and } \gamma := \mathbf{E}_{v_1}^{G_m}[\textstyle\sum_{j=0}^{T_U-1} \mathbf{1}_{\{X_j = v_1\}}],$$

and let $(\xi_i)_{i\geq 1}$ be a family of i.i.d standard normal variables, independent of everything else. Define

$$\widehat{\psi_{\mathcal{G}_n}}(z_1) := \alpha \widehat{\psi_{\mathcal{G}_n}}(y_1) + \beta \widehat{\psi_{\mathcal{G}_n}}(y_k) + \sqrt{\gamma}\xi_1.$$

Then for $i \geq 2$, define recursively

$$\widehat{\psi_{\mathcal{G}_n}}(z_i) := \alpha_{m-(k+i-1)} \widehat{\psi_{\mathcal{G}_n}}(z_{i-1}) + \beta_{m-(k+i-1)} \widehat{\psi_{\mathcal{G}_n}}(y_k) + \sqrt{\gamma_{m-(k+i-1)}}\xi_i \tag{3.86}$$

where we set

$$U_i := G_m \setminus \{u_1, \dots, u_k, v_1, \dots, v_{i-1}\},\ \alpha_{m-(k+i-1)} := \mathbf{P}_{v_i}^{G_m}(X_{T_{U_i}} = v_{i-1},\ T_{U_i} < +\infty),$$

$$\beta_{m-(k+i-1)} := \mathbf{P}_{v_i}^{G_m}(X_{T_{U_i}} = u_k,\ T_{U_i} < +\infty) \text{ and } \gamma_{m-(k+i-1)} := \mathbf{E}_{v_i}^{G_m}[\textstyle\sum_{j=0}^{T_{U_i}-1} \mathbf{1}_{\{X_j = v_i\}}].$$

Case 2: $\widehat{\psi_{\mathcal{G}_n}}$ has not been defined on any vertex of C. There exists a unique path of consecutive vertices $y_1, \dots, y_j$ for some $j \geq 2$ such that $y_j \in C$, and for $2 \leq i \leq j-1$, $\widehat{\psi_{\mathcal{G}_n}}(y_i)$ has not been defined and $y_i \notin C$. Reveal $B_{\mathcal{G}_n}(\{y_1 \dots, y_{j-1}\} \cup C, r_n)$. Give a mark to the vertices of $\{y_1, \dots, y_{j-1}\} \cup C$. Take y_1 away from ∂T_y.

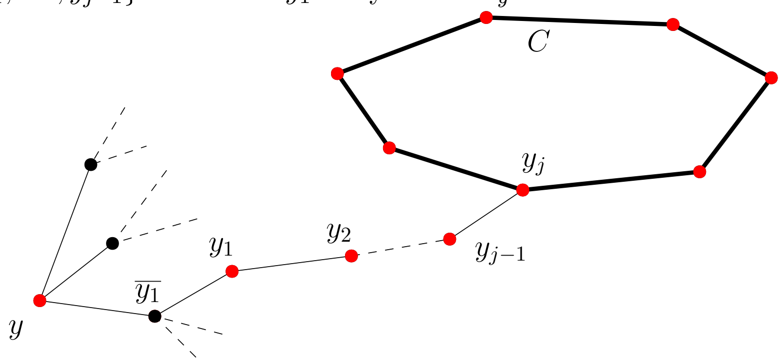

Figure 5. Case 2. Marked vertices are in red.

We define $\widehat{\psi_{\mathcal{G}_n}}$ on $\{y_1, \dots, y_{j-1}\} \cup C$ in a way similar to Case 1. Let $(\xi_i)_{i \geq 1}$ be a sequence of i.i.d. standard normal variables, independent of everything else. For $i = 1, 2, \dots, j$, set

$$\widehat{\psi_{\mathcal{G}_n}}(y_i) := \alpha'_{j-i}\widehat{\psi_{\mathcal{G}_n}}(y_{i-1}) + \sqrt{\gamma'_{j-i}}\xi_i, \tag{3.87}$$

where $\alpha'_{j-i} := \mathbf{P}_z^{G_m}(H_{\{z'\}} < +\infty)$ and $\gamma'_{j-i} := \mathbf{E}_z^{G_m}[\sum_{l=0}^{H_{\{z'\}}-1} \mathbf{1}_{\{X_l = z\}}]$, z, z' being two neighbours in G_m such that z (resp. z') is at distance $j-i$ (resp. $j-i+1$) of C_m. Then, define $\widehat{\psi_{\mathcal{G}_n}}$ on C as in Case 1 with $k = 1$.

After the discovery of C. Resume the subexplorations as before, starting by the subexploration interrupted when C was discovered. For any marked vertex y: when proceeding to the step from y_1 in the subexploration from y, do not reveal the edges of $B_{\mathcal{G}_n}(y_1, \overline{y_1}, r_n + \ell)$ in the direction of any already marked neighbour of y_1.

III - If a second cycle arises, the exploration from x is over, and is not successful.

If at some point,

D1 $\mathcal{M} = \emptyset$,

D2 $\widehat{\psi_{\mathcal{G}_n}}$ has been defined on at most $\lfloor K' \log n \rfloor$ vertices, and

D3 at most one cycle has been discovered,

say that the first phase of the exploration from x is successful. Denote $\mathcal{S}_1(x)$ this event. If all subexploration trees were infertile, then the exploration from x is over, and said to be successful. Denote $\mathcal{S}_{1,\text{stop}}(x)$ this event.

Proposition 3.7.2. *Fix $K, a > 0$. If K' is large enough, then for large enough n and every $x \in V_n$, $\mathbb{P}_{ann}(\mathcal{S}_1(x)) \geq 1 - n^{-5/4}$.*

Proof. In a nutshell, the argument is as follows: by our choice of ℓ and Lemma 3.7.3, the same reasoning as that in the beginning of Proposition 3.3.6 shows that the size of a subexploration tree has a bounded expectation (as $n \to +\infty$) with exponential moments. Hence if $K'' > 0$ is large enough, with probability $\geq 1-n^{-2}$, the first $K'' \log n$ subexplorations encompass less than $K''^2 \log n$ vertices. The total number of vertices seen is at most $K''^2 \log n \times (d-1)^{2r_n} \leq n^{3/10}$. In that case, Lemma 3.2.2 entails that we see at most one cycle, with probability at least $1 - n^{-4/3}$. This allows to control $|\mathcal{M}|$, since a cycle brings less than $2d^{\ell'} r_n = \Theta(\log n)$ marked vertices. Without cycles, each vertex where $\widehat{\psi_{\mathcal{G}_n}}$ is defined has a chance $O(n^{-a})$ to get a mark. Hence for K'' large enough, with probability at least $1 - n^{-5/4}$, less than $K'' \log n$ vertices get a mark in the first $K'' \log n$ explorations, ensuring that the exploration from x is finished - and successful.

Let $\mathcal{E}_{1,n}$ be the event that two cycles are discovered before $n^{3/10}$ vertices have been seen during the exploration from x. By (3.14) with $k = 2$, $m_0 = 1$, $m_E = 0$ and $m \leq n^{3/10}$,

$$\mathbb{P}_{ann}(\mathcal{E}_{1,n}) \leq n^{-4/3}. \tag{3.88}$$

Suppose that we perform the subexploration from some vertex y. Let $y_1 \in \partial T_y$, with $y_1 \neq y$. If no cycle arises when revealing the $(r_n + \ell)$-offspring of y_1, then $|\partial T_y(y_1)| - 1$, the increment of $|\partial T_y|$ during that step, dominates stochastically $\rho_{\ell,h,\delta} - 1$, by definition of $\rho_{\ell,h,\delta}$ at Lemma 3.7.3, as soon as $n^{-a} < \delta$. If a cycle arises (which happens at most once on $\mathcal{E}_{1,n}^c$), at most two vertices are marked and taken away from ∂T_y. After j steps (if the subexploration is not over), $|\partial T_y| \underset{st.}{\geq} S_{j-1} - 2$, where S_{j-1} is the sum of $j-1$ i.i.d. random variables of law $\rho_{\ell,h,\delta} - 1$, hence taking values in the bounded interval $[0, d^{\ell}]$, and with a positive expectation by Lemma 3.7.3 (the -1 in $j-1$ comes from the fact that we do not include the step from y, and the $+2$ from the possibility that two vertices can be taken away from ∂T_y if there is a cycle). Hence

$$\mathbb{P}_{ann}(\text{the subexploration from } y \text{ lasts more than } j \text{ steps}) \leq \mathbb{P}(S_{j-1} \leq K \log n + 2)$$

By the exponential Markov inequality, there exist constants $c, c' > 0$ such that

$$\text{for every } j \geq 2K(\mathbb{E}[\rho_{\ell,h,\delta}] - 1)^{-1} \log n, \quad \mathbb{P}(S_{j-1} \leq K \log n + 2) \leq ce^{-c'j}.$$

Now, let $K'' > 0$ and focus on the first $\lfloor K'' \log n \rfloor$ subexplorations (or on all subexplorations if there are less than $\lfloor K'' \log n \rfloor$ of them). Let N be the total number of steps during those subexplorations. On $\mathcal{E}_{1,n}^c$, N is stochastically dominated by a sum S of $\lfloor K'' \log n \rfloor$ i.i.d. variables of some law μ (independent of n) such that

$$\text{for every } j \geq 2K(\mathbb{E}[\rho_{\ell,h,\delta}] - 1)^{-1} \log n, \quad \mu([j, +\infty)) \leq ce^{-c'j}.$$

Hence, letting $\mathcal{E}_{2,n} := \{N \geq K''^2 \log n\}$, by the exponential Markov inequality,

$$\mathbb{P}_{ann}(\mathcal{E}_{2,n} \cap \mathcal{E}_{1,n}^c) \leq \mathbb{P}_{ann}(S \geq K'' \lfloor K'' \log n \rfloor) \leq (\mathbb{E}[e^{c'Y/2}] e^{-c'K''\mu/2})^{\lfloor K'' \log n \rfloor}$$

for large enough n and for $Y \sim \mu$. Taking K'' large enough, we have for n large enough:

$$\mathbb{P}_{ann}(\mathcal{E}_{2,n} \cap \mathcal{E}_{1,n}^c) \leq n^{-2}. \tag{3.89}$$

In a step of a subexploration, less than $(d-1)^{2r_n} \leq n^{1/10}$ new vertices are seen. When a cycle C is revealed, there are at most $2r_n$ new vertices in C (and on the path leading to C, in Case 2), so that less than $3r_n(d-1)^{r_n} \leq n^{1/10}$ new vertices are seen. Therefore, on $\mathcal{E}_{1,n}^c \cap \mathcal{E}_{2,n}^c$, less than $n^{3/10}$ vertices are seen during the first $\lfloor K'' \log n \rfloor$ subexplorations.

Moreover, in a step of a subexploration, less than d^ℓ vertices are added to the subexploration tree. Hence if $K' > K''^2 d^\ell$, on $\mathcal{E}_{2,n}^c$, D2 holds.

We now estimate the total number of vertices that will receive a mark. When a cycle C appears, our construction implies that less than $3r_n$ vertices receive a mark. When performing m steps in a subexploration, the number of marked vertices obtained is stochastically dominated by a binomial random variable $\mathrm{Bin}(m, d^\ell n^{-a})$. Indeed, at each step, we reveal $\widehat{\psi_{\mathcal{G}_n}}$ (and thus $\varphi_{\mathbb{T}_d}$) on less than d^ℓ vertices. And for any vertex $y \in \mathbb{T}_d \setminus \{\circ\}$, by Proposition 3.3.1,

$$\max_{a' \geq h} \mathbb{P}^{\mathbb{T}_d}(\varphi_{\mathbb{T}_d}(y) \in [h - n^{-a}, h + n^{-a}] \,|\, \varphi_{\mathbb{T}_d}(\overline{y}) = a') \leq \frac{2n^{-a}}{\sqrt{2\pi d/(d-1)}} \leq n^{-a}.$$

Hence, if $\mathcal{E}_{3,n}$ is the event that more than $3r_n + \frac{K''}{2} \log n$ marks are given during the first $\lfloor K'' \log n \rfloor$ subexplorations and if $Z \sim \mathrm{Bin}(\lceil K''^2 \log n \rceil, C_2 d^\ell n^{-a})$,

$$\mathbb{P}_{ann}(\mathcal{E}_{1,n}^c \cap \mathcal{E}_{2,n}^c \cap \mathcal{E}_{3,n}) \leq \mathbb{P}\left(Z \geq \frac{K''}{2} \log n\right) \leq \binom{\lceil K''^2 \log n \rceil}{\frac{K''}{2} \log n} (C_2 d^\ell n^{-a})^{\frac{K''}{2} \log n},$$

thus by(3.12), for large enough n,

$$\mathbb{P}_{ann}(\mathcal{E}_{1,n}^c \cap \mathcal{E}_{2,n}^c \cap \mathcal{E}_{3,n}) \leq \left(\lceil K''^2 \log n \rceil C_2 d^\ell n^{-a}\right)^{\frac{K''}{2} \log n} \leq n^{-2}. \tag{3.90}$$

Taking $K'' > 1$, on $\mathcal{E}_{1,n}^c \cap \mathcal{E}_{2,n}^c \cap \mathcal{E}_{3,n}^c$, less than $1 + 3r_n + \frac{K''}{2} \log n \leq \lfloor K'' \log n \rfloor$ vertices receive a mark during the first $\lfloor K'' \log n \rfloor$ subexplorations, so that $\mathcal{M} = \emptyset$ after at most $\lfloor K'' \log n \rfloor$ subexplorations. Therefore, on $\mathcal{E}_{1,n}^c \cap \mathcal{E}_{2,n}^c \cap \mathcal{E}_{3,n}^c$, conditions D1, D2 and D3 hold. By (3.88), (3.89) and (3.90), for large enough n

$$\mathbb{P}_{ann}(\mathcal{E}_{1,n}^c \cap \mathcal{E}_{2,n}^c \cap \mathcal{E}_{3,n}^c) \geq 1 - 2n^{-2} - n^{-4/3} \geq 1 - n^{-5/4},$$

and this concludes the proof. □

We state here the variant of Lemma 3.3.9. Let $\delta \in [0, h_\star - h)$. For $\ell \geq 1$, write $\mathcal{Z}_\ell$ for the ℓ-th generation of the connected component of $\circ$ in $(\{\circ\} \cup E_{\varphi_{\mathbb{T}_d}}^{\geq h+\delta}) \cap \mathbb{T}_d^+$. Let $\rho_{\ell,h,\delta}$ be the law of $|\mathcal{Z}_\ell|$ conditionally on $\varphi_{\mathbb{T}_d}(\circ) = h$, so that $\rho_{\ell,h} = \rho_{\ell,h,0}$ (recall (3.28)). The following result is a straightforward consequence of Lemma 3.3.9 (applied to $h + \delta$ instead of h) and from the fact that $\mathbb{P}_h^{\mathbb{T}_d}(\exists v \in \partial B(\circ, 1), \varphi_{\mathbb{T}_d}(v) \geq h + \delta) > 0$.

Lemma 3.7.3. *For every $\delta \in [0, h_\star - h)$, if ℓ is large enough,*

$$\mathbb{E}[\rho_{\ell,h,\delta}] > 1.$$

3.7.3 Second phase

If $\mathcal{S}_{1,\text{stop}}(x)$ holds, i.e. all subexploration trees are infertile, the exploration is over. In this Section, we suppose that $\mathcal{S}_1(x)\backslash\mathcal{S}_{1,\text{stop}}(x)$ holds. For every fertile tree, we perform an exploration similar to that of Section 3.5.1 from each vertex of its boundary, and show that with probability at least $1-n^{-6/5}$, at least one exploration per fertile tree is successful, and hence has a boundary of size $\Theta(n^{1/2}b_n)$ (Proposition 3.7.4). We illustrate this second phase in Figure 6 below.

Let $T_1, \dots T_m$ be the fertile subexploration trees for some positive integer m. For every $q \in \{1, \dots, m\}$, denote $y_{q,1}, y_{q,2}, \dots$ the vertices of ∂T_q. For $q = 1, 2, \dots$ successively, we perform the explorations from $y_{q,i}$, $i \geq 1$ as defined in Section 3.5.1 (using an array of independent standard normal variables $(\xi_{y_{q,i},k,j})_{k,j\geq 0}$). If y is the j-th vertex of the k-th generation in the exploration tree of $y_{q,i}$, let recursively

$$\widehat{\psi_{\mathcal{G}_n}}(y) := \frac{\widehat{\psi_{\mathcal{G}_n}}(\overline{y})}{d-1} + \sqrt{\frac{d}{d-1}}\xi_{y_{q,i},k,j},$$

so that $\widehat{\psi_{\mathcal{G}_n}}$ plays the role of $\varphi_{\mathbb{T}_d}$ on $\mathbb{T}_d$ in Section 3.5.1. We implement two modifications:

- we do not explore towards $\overline{y_{q,i}}$, the parent of $y_{q,i}$ in T_q (hence we identify $\widehat{\psi_{\mathcal{G}_n}}$ to the GFF on a subtree of $\mathbb{T}_d^+$ instead of $\mathbb{T}_d$), and
- we do not stop the exploration if $\widehat{\psi_{\mathcal{G}_n}}(y_{q,i}) < h + \log^{-1} n$ (we only know *a priori* from the first phase that $\widehat{\psi_{\mathcal{G}_n}}(y_{q,i}) \geq h + n^{-a}$), and
- we stop the exploration if it meets a vertex already discovered in the first phase or during the previous exploration of some $y_{q',j}$ (thus with $q' < q$, or $q' = q$ and $j < i$).

A vertex $y_{q,i}$ whose exploration is successful is **back-spoiled** if one vertex of its exploration is seen later during the exploration of $y_{q',j}$. Let

$$\mathcal{E}_{4,n} := \{\exists q \leq m, \text{ all successful explorations of } y_{q,1}, y_{q,2}, \dots, \text{ are back-spoiled}\}$$

On $\mathcal{S}_2(x) := (\mathcal{S}_1(x) \setminus \mathcal{S}_{1,\text{stop}}(x)) \cap \mathcal{E}_{4,n}^c$, say that the second phase is successful.

Proposition 3.7.4. *If K of Proposition 3.7.2 is large enough, then for n large enough and every $x \in V_n$,*

$$\mathbb{P}_{ann}(\mathcal{S}_2(x) \cup \mathcal{S}_{1,\text{stop}}(x)) \geq 1 - n^{-6/5} \tag{3.91}$$

Proof. Say that $y_{q,i}$ is **spoiled** if it is met during the previous exploration of some $y_{q',j}$. Define $\mathcal{E}_{5,n} := \{\text{at least } \lfloor 1000 \log n \rfloor \text{ vertices are spoiled or back-spoiled}\}$. We claim that for n large enough,

$$\mathbb{P}_{ann}(\mathcal{E}_{5,n}) \leq n^{-2}, \tag{3.92}$$

and for some constant $K_{10} > 0$ (that only depends on d and h), for each of these vertices $y_{q,i}$,

$$\mathbb{P}_{ann}(\text{the exploration from } y_{q,i} \text{ is successful} \mid y_{q,i} \text{ is not spoiled}) \geq K_{10}. \tag{3.93}$$

On $\mathcal{E}_{5,n}^c$, if (3.93) holds, there are at least $(K-1000)\log n$ non-spoiled vertices on each ∂T_q (and at most $(K+1)\log n$ since each step of a subexploration brings less than $\log n$ vertices to ∂T_q), where K was defined in the beginning of Section 3.7.2. And, if more than $\lfloor 1000\log n\rfloor$ vertices of each ∂T_q are successful, one of them will be successful and not back-spoiled, thus fulfilling the requirement of $\mathcal{S}_2(x)$.

If K is large enough, for n large enough, the probability that no more than $\lfloor 1000\log n\rfloor$ explorations from ∂T_q are successful is at most

$$\binom{\lfloor (K+1)\log n\rfloor}{\lfloor 1000\log n\rfloor}(1-K_{10})^{(K-1000)\log n} \le n\left(\frac{(K+1)^{K+1}}{1000^{1000}(K-999)^{K-999}}(1-K_{10})^{K-1000}\right)^{\log n} = o(n^{-3})$$

by Stirling's formula. Hence by a union bound on $1\le q\le m$, noticing that $m\le 2K'/K$ for n large enough by D2, we have:

$$\mathbb{P}_{ann}((\mathcal{S}_2(x)\cup\mathcal{S}_{1,\text{stop}}(x))^c) \le \mathbb{P}_{ann}(\mathcal{S}_1(x)^c)+\mathbb{P}_{ann}(\mathcal{E}_{5,n}^c)+mn^{-3}\le 2n^{-5/4}$$

by Proposition 3.7.2 and (3.92), and this concludes the proof. Hence, it remains to establish (3.92) and (3.93).

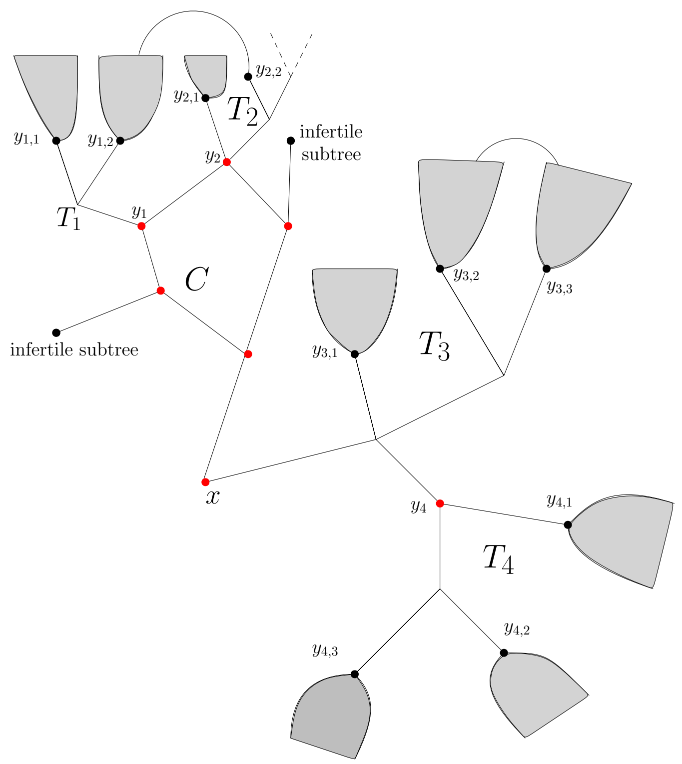

Figure 6. Marked vertices are in red. There are four fertile subexploration trees (T_1 rooted at y_1, T_2 rooted at y_2, T_3 rooted at x and T_4 rooted at y_4), and two infertile ones. Lightgray areas correspond to the explorations of the $y_{q,i}$'s in the second phase. $y_{2,2}$ is spoiled by $y_{1,2}$. $y_{3,2}$ is back-spoiled by $y_{3,3}$. Each of the subxploration trees will be either included in $\mathcal{C}_x^{\mathcal{G}_n,h}$ or have no common vertex with $\mathcal{C}_x^{\mathcal{G}_n,h}$, depending on the value of $\psi_{\mathcal{G}_n}$ on the marked vertices.

Proof of (3.92). Note that by D2, by Remark 3.5.2 and by (3.84),

$$\text{less than } n^{1/2}\log^{-1} n \text{ vertices and half-edges have been seen in the first two phases.} \quad (3.94)$$

In particular, less than $n^{1/2}\log^{-1} n$ edges are built during second phase. Since there are at most $K'\log n$ $y_{q,i}$'s by D2, each new edge has a probability at most

$$K'\log n/(n - n^{1/2}\log^{-1} n) \leq 2K'n^{-1}\log n$$

to spoil a vertex. Thus, the number of spoiled vertices is stochastically dominated by a random variable $Z \sim \text{Bin}(n^{1/2}\log^{-1} n, 2K'n^{-1}\log n)$. For n large enough,

$$\mathbb{P}(Z \geq 10) \leq \binom{n^{1/2}\log^{-1} n}{10}\left(\frac{2K'\log n}{n}\right)^{10} \leq n^5\frac{\log^{20} n}{n^{10}} \leq n^{-3}.$$

Moreover, by (3.15) with $k = \lfloor 999\log n\rfloor$ and $m_0, m_1, m_E, m \leq n^{1/2}\log^{-1} n$ due to (3.94),

$$\mathbb{P}_{ann}(\text{more than } \lfloor 999\log n\rfloor \text{ vertices are back-spoiled}) \leq n^{-3}.$$

(3.92) follows.

Proof of (3.93). By (3.94) and (3.14) with $k = 1$ and $m_0, m_1, m_E, m \leq n^{1/2}\log^{-1} n$, for n large enough,

$$\mathbb{P}_{ann}(\text{a cycle is created during the exploration from } y_{q,i}) \leq \log^{-1} n.$$

The law of $\widehat{\psi_{\mathcal{G}_n}}$ on the exploration tree from $y_{q,i}$ is that of $\varphi_{\mathbb{T}_d}$ on an isomorphical subtree of $\mathbb{T}_d^+$ (and not $\mathbb{T}_d$, since we do not explore towards $\overline{y_{q,i}}$), with $\varphi_{\mathbb{T}_d}(\circ) = \widehat{\psi_{\mathcal{G}_n}}(y_{q,i})$. Denote $\mathcal{C}_\circ^n$ the connected component of $\circ$ in $E_{\varphi_{\mathbb{T}_d}}^{\geq h+\log^{-1} n,+} \cup \{\circ\}$, and $\mathcal{Z}_k$ its k-th generation for every $k \geq 0$. Then

$$\mathbb{P}_{ann}(\text{the exploration from } y_{q,i} \text{ is successful}|y_{q,i} \text{ is not spoiled}) \geq p_n - \log^{-1} n,$$

where $p_n := \min_{b\geq h+n^{-a}} \mathbb{P}_b^{\mathbb{T}_d}(\exists k \leq \log_{\lambda_h} n, |\mathcal{Z}_k| \geq n^{1/2}b_n)$ (recall that $\widehat{\psi_{\mathcal{G}_n}}(y_{q,i}) \geq h + n^{-a}$). Let $\delta \in (0, h_\star - h)$. Clearly, there exists $p' > 0$ such that for n large enough,

$$\min_{b\geq h+n^{-a}} \mathbb{P}_b^{\mathbb{T}_d}(\exists v \in \mathcal{Z}_1,\ \varphi_{\mathbb{T}_d}(v) \geq h+\delta) > p'.$$

For $\varepsilon > 0$ small enough so that $\log_{d-1}(\lambda_{h+\delta} - \varepsilon) \geq (3\log_{d-1}\lambda_h)/4$ (such ε exists by continuity of $h' \mapsto \lambda_{h'}$, Proposition 3.3.4), for $n \in \mathbb{N}$,

$$\begin{aligned} p_n'' &:= \min_{b\geq h+\delta} \mathbb{P}_b^{\mathbb{T}_d}(\exists k \leq \log_{\lambda_h} n - 1,\ |\mathcal{Z}_k^{h+\log^{-1} n,+}| \geq n^{1/2}b_n) \geq \min_{b\geq h+\delta} \mathbb{P}_b^{\mathbb{T}_d}(|\mathcal{Z}_{\lfloor\log_{\lambda_h} n\rfloor - 1}^{h+\delta,+}| \geq n^{1/2}b_n) \\ &\geq \min_{b\geq h+\delta} \mathbb{P}_b^{\mathbb{T}_d}(|\mathcal{Z}_{\lfloor\log_{\lambda_h} n\rfloor - 1}^{h+\delta,+}| \geq (\lambda_{h+\delta} - \varepsilon)^{\lfloor\log_{\lambda_h} n\rfloor - 1}). \end{aligned}$$

By Proposition 3.3.8, $\liminf_{n\to+\infty} p_n'' =: p'' > 0$. Since $p_n \geq p'p_n''$ for all $n \geq 1$,

$$\mathbb{P}_{ann}(\text{the exploration from } y_{q,i} \text{ is successful}|y_{q,i} \text{ is not spoiled}) \geq p_n - \log^{-1} n \geq \tfrac{p'p''}{2}$$

for n large enough, and we can take $K_{10} = \frac{p'p''}{2}$. This shows (3.93). □

3.7.4 Third phase

Suppose now that we are on $\mathcal{S}_2(x)$. For $1 \leq q \leq m$, denote y_q one vertex of ∂T_q whose exploration was successful and not back-spoiled in the second phase, T_{y_q} its exploration tree, and B_q the boundary of T_{y_q}. In this section, we connect each B_q to $\Theta(n)$ vertices in a fashion similar to Section 3.6.1. However, revealing the GFF on a positive proportion of the vertices would prevent us to use an approximation of the GFF as in Proposition 3.4.1.
To circumvent this difficulty, denote $R_{1,2}$ the set of vertices seen in the first two phases, and partition $V_n \setminus R_{1,2}$ arbitrarily in sets $D_1, D_2, \ldots, D_r$ for some $r \in \mathbb{N}$ such that

$$|D_1| = |D_2| = \ldots = |D_{r-1}| = \lfloor K_{11} \log n \rfloor, \tag{3.95}$$

for some constant $K_{11} > 0$. We will connect each B_q to a positive proportion of the vertices of D_1 only, with $\mathbb{P}_{ann}$-probability $1 - n^{-3}$ (Proposition 3.7.5), before revealing the GFF on the vertices of the first two phases, and on the connection from D_1 to B_q in Section 3.7.5. The result will follow by symmetry of the D_i's and a union bound on $1 \leq i \leq r-1$.

The exploration.
1) The w-explorations. Let $w_1, \ldots, w_{\lfloor K_{11} \log n \rfloor}$ be the vertices of D_1. We proceed successively to the **w-explorations** of $w_1, w_2, \ldots$, i.e. for $i \geq 1$, we perform the exploration from w_i as in Section 3.5.1, but stop it if we reach a vertex seen in the first two phases or in the w-exploration of some w_j, $j \leq i-1$. In particular, if w_i was discovered during the exploration of some $w_j, j \leq i-1$, say then that w_i is **w-spoiled** and do not proceed to its w-exploration. Denote R_w the set of vertices seen during all the w-explorations.
For $i \geq 1$, if we explore w_i and C2 happens, say that the w-exploration from w_i is **w-successful**. Let s_0 be the number of w-successful vertices. Let $w_{i_1}, \ldots, w_{i_{s_0}}$ be the w-successful vertices with $i_1 < \ldots < i_{s_0}$. Let $T_{w_{i_j}}$ be the exploration tree of w_{i_j}, for $j \in \{1, \ldots, s_0\}$. Take away

- from each $\partial T_{w_{i_j}}$: the vertices that are seen in the w-exploration of some w_i, for $i > i_j$,
- from each B_q: the vertices z such that $B_{\mathcal{G}_n}(z, \overline{z}, a_n)$ intersects R_w.

Say that those vertices are **w-back-spoiled**.
2) The joining balls. For $q = 1, \ldots, m$ successively, we develop balls from B_q to the $\partial T_{w_{i_j}}$'s, $1 \leq j \leq s_0$, with a few modifications w.r.t the construction of Section 3.6.1: let $z_{1,q}, z_{2,q}, \ldots$ be the vertices of B_q. For $z_{i,q} \in B_q$, let

$$B^*_{i,q} := \cup_{(i',q'):q'<q \text{ or } q'=q, i'<i} B^*(z_{i',q'}, a'_n),$$

and let $R_{i,q} := R_{1,2} \cup R_w \cup B^*_{i,q}$ be the vertices seen before building $B^*(z_i, a'_n)$.
Replace B^*_j, Q_j and $B_{\mathcal{G}_n}(T_y, a_n)$ of Section 3.6.1 by $B^*_{i,q}$, $R_{i,q}$ and $\cup_{j=1}^{s_0} B_{\mathcal{G}_n}(T_{w_{i_j}}, a_n)$ respectively. Say that $B^*(z_{i,q}, a'_n)$ is a **J-joining ball** if it hits $B_{\mathcal{G}_n}(T_{w_{i_J}}, a_n)$ at one vertex after $a'_n - 2a_n$ steps, and no other intersection with vertices seen previously is created.

Proposition 3.7.5. *For n large enough, we have*

$$\mathbb{P}_{ann}(\mathcal{S}_2(x)\cap\{\exists(J,q),\ \text{there are less than } \log^{\gamma-3\kappa-18} n\ J\text{-joining balls from } B_q\}) \leq n^{-4} \quad (3.96)$$

and if K_{11} is large enough, then for large enough n:

$$\mathbb{P}_{ann}(\mathcal{S}_2(x)\cap\{s_0 \leq \log n\}) \leq n^{-4}. \quad (3.97)$$

Proof. **Proof of (3.96).** We adapt the proof of Lemma 3.6.1. Since there are less than $\log^2 n$ w_i's (and $|R_{1,2}|$ is controlled by (3.94)), we can replace (3.61) by

$$\text{less than } n^{1/2}\log^{\gamma+2} n \text{ vertices are seen during the three phases.} \quad (3.98)$$

Let $B^* := \cup_{q=1}^m \cup_{z\in B_q} B^*(z,a'_n)$. (3.63) becomes

$$\mathbb{P}_{ann}(\mathcal{S}_2(x)\cap\{\,|B^*\cap(\cup_{j=1}^{s_0} B_{\mathcal{G}_n}(T_{z_{i_j}},a_n))| \geq \log^{3\gamma} n\}) \leq n^{-5} \quad (3.99)$$

Let N be the number of vertices of $\cup_{q=1}^m B_q$ that are spoiled, i.e. the vertices z such that $B^*(z,a_n) = B_{\mathcal{G}_n}(z,\overline{z},a_n)$ is hit by a previously constructed $B^*(z',a'_n)$. (3.64) becomes

$$\mathbb{P}_{ann}(\mathcal{S}_2(x)\cap\{N\geq \log^{3\gamma} n\}) \leq n^{-5}. \quad (3.100)$$

In addition, by (3.15) with $k=\log^2 n$, $m_0=1$, $m_1, m_E, m \leq n^{1/2}\log^{-1/2} n$,

$$\mathbb{P}_{ann}\left(\mathcal{E}_{6,n}\right) \leq n^{-5}, \quad (3.101)$$

where $\mathcal{E}_{6,n} := \mathcal{S}_2(x)\cap\big\{\text{more than } \log^2 n \text{ vertices are } w\text{-back-spoiled}\big\}$. Then, define

$$\mathcal{S}_{i,q} := \mathcal{S}_2(x)\cap\mathcal{E}^c_{6,n}\cap\{|(\cup_{j=1}^{s_0} B_{\mathcal{G}_n}(T_{w_{i_j}},a_n))\cap B^*_{i,q}| \leq \log^{3\gamma} n\}\cap\{z_{i,q} \text{ is not spoiled}\}.$$

It is straightforward to adapt the proof of (3.66) to get that for every $1\leq J\leq s_0$, the probability that $B^*(z_{i,q},a'_n)$ is a J-joining ball, conditionally on $\mathcal{S}_{i,q}$, is at least $n^{-1/2}\log^{\gamma-2\kappa-10} n$. On $\mathcal{S}_2(x)\cap\{N\leq\log^{3\gamma} n\}\cap\mathcal{E}^c_{6,n}$, at least

$$n^{1/2}b_n - \log^2 n - \log^{3\gamma} n \geq n^{1/2}\lfloor \log^{-\kappa-7} n\rfloor$$

$z_{i,q}$'s are neither spoiled nor w-back-spoiled by (3.46). As in the end of the proof of Lemma 3.6.1, if $\gamma > 3\kappa+18$, we get that for n large enough: for every $(J,q)\in(\{1,\dots,s_0\}\cap\{1,\dots,m\})$ and $Z\sim \mathrm{Bin}(n^{1/2}\lfloor\log^{-\kappa-7} n\rfloor, n^{-1/2}\log^{\gamma-2\kappa-10} n)$:

$$\begin{aligned}
&\mathbb{P}_{ann}(\mathcal{S}_2(x)\cap\{\text{there are less than } \log^{\gamma-3\kappa-18} n\ J\text{-joining balls from } B_q\})\\
&\leq \mathbb{P}_{ann}\Big(\mathcal{S}_2(x)\cap(\{\,|B^*\cap(\cup_{j=1}^{s_0} B_{\mathcal{G}_n}(T_{z_{i_j}},a_n))| \geq \log^{3\gamma} n\}\cup\{N\geq\log^{3\gamma} n\}\cup\mathcal{E}_{6,n})\Big)\\
&+\mathbb{P}(Z\leq \log^{\gamma-3\kappa-18} n)\\
&\leq 3n^{-5} + n\max_{0\leq k\leq \log^{\gamma-3\kappa-18} n}\mathbb{P}(Z=k) \quad \text{by (3.99), (3.100) and (3.101)}\\
&\leq 4n^{-5}.
\end{aligned}$$

Since $s_0 \le |D_1| \le \log^2 n$ by (3.95) and $m \le K' \log n$ by D2, a union bound on (J, q) yields (3.96).

Proof of (3.97). We now estimate the probability that at most $\log_n$ w-explorations are w-successful. Note that by Remark 3.5.2 and (3.94),

$$|R_{1,2}| + |R_w| \le n^{1/2} \log^{-1/2} n. \tag{3.102}$$

Hence if $C > 0$ is large enough, by (3.15) with $k = \lfloor C \log n \rfloor$ and $m_0, m_1, m_E, m \le n^{1/2} \log^{-1/2} n$,

$$\mathbb{P}_{ann} \left(\mathcal{S}_2(x) \cap \{\text{more than } C \log n \ w_i\text{'s are } w\text{-spoiled}\} \right) \le n^{-5}. \tag{3.103}$$

Moreover, for every $i \ge 1$, conditionally on the fact that w_i is not w-spoiled, the probability that the w-exploration from w_i is stopped because it reaches a vertex of $R_{1,2}$ or a vertex seen in the exploration of some w_j, $j < i$ is $o(1)$ by (3.14) with $k = 1$, $m_0 = 1$, $m_1, m_E, m \le n^{1/2} \log^{-1/2} n$. Hence, a straightforward adaptation of the proof of (3.54) yields

$$\mathbb{P}_{ann}(\text{the exploration from } w_i \text{ is } w\text{-successful} \,|\, \mathcal{S}_2(x) \cap \{w_i \text{ is not } w\text{-spoiled}\}) \ge \eta(h)/2. \tag{3.104}$$

Take $K_{11} > 3C$. By (3.103), (3.104) and (3.95), if $Z \sim \text{Bin}(\lfloor (K_{11} \log n)/2 \rfloor, \eta(h)/2)$ and n is large enough,

$$\mathbb{P}_{ann}(\mathcal{S}_2(x) \cap \{s_0 \le \log n\}) \le n^{-5} + \mathbb{P}(Z \le \log n).$$

One checks easily that if K_{11} is large enough, then for large enough n,

$$\mathbb{P}(Z \le \log n) \le n^{-5},$$

and (3.97) follows.

□

3.7.5 Revealing $\psi_{\mathcal{G}_n}$ on the three phases

Let R_1^ψ (resp. R_2^ψ) be the set of vertices where $\widehat{\psi_{\mathcal{G}_n}}$ has been defined during the first (resp. second) phase, and R_3^ψ be the set of vertices in the w-successful w-explorations and on the J-joining balls, for all $1 \le J \le m$, on which we will realize $\psi_{\mathcal{G}_n}$ on the third phase.
By Proposition 3.2.4, we can realize $\psi_{\mathcal{G}_n}$ jointly with $\mathcal{G}_n$ by

- proceeding to the three phases of the exploration from x,
- revealing the remaining pairings of half-edges of the $\mathcal{G}_n$,
- defining $\psi_{\mathcal{G}_n}$ on $R_1^\psi \cup R_2^\psi$, in the same order as $\widehat{\psi_{\mathcal{G}_n}}$ has been defined, using the same standard normal variables: we let

$$\psi_{\mathcal{G}_n}(x) := \widehat{\psi_{\mathcal{G}_n}}(x) \sqrt{\tfrac{d-2}{d-1}} \sqrt{G_{\mathcal{G}_n}(x,x)},$$

and for every y, we let $A_y := \{z \in V_n,\ \widehat{\psi_{\mathcal{G}_n}}(z)$ was defined before $\widehat{\psi_{\mathcal{G}_n}}(y)\}$ and

$$\psi_{\mathcal{G}_n}(y) := \mathbb{E}^{\mathcal{G}_n}[\psi_{\mathcal{G}_n}(y)|\sigma(A_y)] + \xi_y\sqrt{\mathrm{Var}(\psi_{\mathcal{G}_n}(y)|\sigma(A_y))},$$

for every $y \in R_1^{\psi}$, where ξ_y is the normal variable used when defining $\widehat{\psi_{\mathcal{G}_n}}(y)$,

- revealing $\psi_{\mathcal{G}_n}$ on R_3^{ψ}, and finally on $V_n \setminus (R_1^{\psi} \cup R_2^{\psi} \cup R_3^{\psi})$.

Let $\mathcal{E}_{7,n} := \{\mathcal{G}_n$ is a good graph$\} \cap \{\max_{z\in V_n} |\psi_{\mathcal{G}_n}(z)| \le \log^{2/3} n\}$,
$\mathcal{S}_1^{\psi}(x) := \mathcal{S}_1(x) \cap \{\sup_{y\in R_1^{\psi}} |\psi_{\mathcal{G}_n}(y) - \widehat{\psi_{\mathcal{G}_n}}(y)| \le n^{-a}/2\}$,
$\mathcal{S}_2^{\psi}(x) := \mathcal{S}_2(x) \cap \mathcal{S}_1^{\psi}(x) \cap \{\sup_{y\in R_2^{\psi}} |\psi_{\mathcal{G}_n}(y) - \widehat{\psi_{\mathcal{G}_n}}(y)| \le (\log^{-1} n)/2\}$, and
$\mathcal{S}_{3,i}^{\psi}(x) := \mathcal{S}_2^{\psi}(x) \cap \{\forall q \in \{1,\dots,m\}$ at least $\log n$ vertices of D_i are connected to T_q in $E_{\psi_{\mathcal{G}_n}}^{\ge h}\}$ for every $i \ge 1$.
Suppose that for $a > 0$ (defined in the beginning of Section 3.7.2) small enough, and for n large enough:

$$\mathbb{P}_{ann}((\mathcal{S}_1(x) \setminus \mathcal{S}_1^{\psi}(x)) \cap \mathcal{E}_{7,n}) \le n^{-3}, \tag{3.105}$$

$$\mathbb{P}_{ann}((\mathcal{S}_2(x) \setminus \mathcal{S}_2^{\psi}(x)) \cap \mathcal{E}_{7,n}) \le n^{-3}, \tag{3.106}$$

and for every $1 \le i \le r-1$,

$$\mathbb{P}_{ann}((\mathcal{S}_2(x) \setminus \mathcal{S}_{3,i}^{\psi}(x)) \cap \mathcal{E}_{7,n}) \le n^{-3}. \tag{3.107}$$

Letting $\mathcal{S}_{1,\mathrm{stop}}^{\psi}(x) := \mathcal{S}_{1,\mathrm{stop}}(x) \cap \mathcal{S}_1^{\psi}(x)$, (3.105), (3.106), 3.107) and (3.91) imply that

$$\mathbb{P}_{ann}((\mathcal{S}_{1,\mathrm{stop}}^{\psi}(x) \cup (\cap_{i=1}^{r-1}\mathcal{S}_{3,i}^{\psi}(x))\,) \cap \mathcal{E}_{7,n}) \ge 1 - n^{-7/6}. \tag{3.108}$$

On $\mathcal{S}_{1,\mathrm{stop}}^{\psi}(x) \cup (\cap_{i=1}^{r-1}\mathcal{S}_{3,i}^{\psi}(x))$, we have the following alternative:

- either $\mathcal{C}_x^{\mathcal{G}_n,h}$ contains a subexploration tree T_q whose exploration was fertile, the exploration from y_q is successful and connected to at least $\log n$ vertices of every D_i, $1 \le i \le r-1$ in $E_{\psi_{\mathcal{G}_n}}^{\ge h}$;
- or $\mathcal{C}_x^{\mathcal{G}_n,h}$ contains no such tree, and $\mathcal{C}_x^{\mathcal{G}_n,h} \subseteq R_1^{\psi}$, so that $|\mathcal{C}_x^{\mathcal{G}_n,h}| \le K_0 \log n$, where we take $K_0 \ge K'$, and where K' is the constant of Proposition 3.7.2.

Note that the second case comprises $\mathcal{S}_{1,\mathrm{stop}}^{\psi}(x)$ but is a priori not included in it: there could exist fertile subexploration trees not connected to x in $E_{\psi_{\mathcal{G}_n}}^{\ge h}$ if $\psi_{\mathcal{G}_n}$ is below h on the appropriate marked vertices.
In the first case, $\mathcal{C}_x^{\mathcal{G}_n,h}$ contains at least $\log n$ vertices of each D_i, $1 \le i \le r-1$, so that by (3.94) and (3.95) for n large enough:

$$|\mathcal{C}_x^{\mathcal{G}_n,h}| \ge (r-1)\log n \ge \log n\ \frac{n - |R_{1,2}| - |D_r|}{K_{11}\log n} \ge \frac{n}{2K_{11}}.$$

Letting thus $K_{12} := (2K_{11})^{-1}$, for every n large enough, we have on $\mathcal{S}^{\psi}_{1,\text{stop}}(x) \cup (\cap_{i=1}^{r-1} \mathcal{S}^{\psi}_{3,i}(x))$:

$$|\mathcal{C}_x^{\mathcal{G}_n,h}| \le K_0 \log n \text{ or } |\mathcal{C}_x^{\mathcal{G}_n,h}| \ge K_{12} n.$$

By (3.108) and a union bound on all $x \in V_n$,

$$\mathbb{P}_{ann}(K_0 \log n \le |\mathcal{C}_2^{(n)}| \le K_{12} n) \le n^{-1/6} + \mathbb{P}_{ann}(\mathcal{E}^c_{7,n}).$$

By Remark 3.6.5, $|\mathcal{C}_2^{(n)}| \le K_{12} n$ $\mathbb{P}_{ann}$-w.h.p. By Proposition 3.2.1 and Lemma 3.2.5, we have $\mathbb{P}_{ann}(\mathcal{E}^c_{7,n}) \to 0$ so that

$$\mathbb{P}_{ann}(|\mathcal{C}_2^{(n)}| \le K_0 \log n) \to 1,$$

yielding (3.3). Hence, it remains to show (3.105), (3.106) and 3.107).

The field $\psi_{\mathcal{G}_n}$ on the first phase: proof of (3.105)

Proposition 3.7.6. *Let a, K_0, ℓ be such that Proposition 3.7.2 holds with $K' = K_0$, and such that the conclusion of Lemmas 3.7.7 and 3.7.8 hold. Then for n large enough, (3.105) holds.*

To prove Proposition 3.7.6, we need two variants of Proposition 3.4.1. The first consists in replacing the "security radius" $a_n = \Theta(\log\log n)$ by $r_n = \Theta(\log n)$.

Lemma 3.7.7. *If $a > 0$ is small enough, then the following holds for n large enough. Assume that $\mathcal{G}_n$ is a good graph, that $A \subseteq V_n$ satisfies*

- $|A| \le n^{2/3}$,
- $\text{tx}(B_{\mathcal{G}_n}(A, r_n)) = \text{tx}(A)$, *and*
- $\max_{z\in A} |\psi_{\mathcal{G}_n}(z)| \le \log^{2/3} n$.

For every $y \in \partial B_{\mathcal{G}_n}(A, 1)$, writing $\overline{y}$ for the unique neighbour of y in A, we have:

$$\left| \mathbb{E}^{\mathcal{G}_n}[\psi_{\mathcal{G}_n}(y) | \sigma(A_y)] - \frac{1}{d-1} \psi_{\mathcal{G}_n}(\overline{y}) \right| \le n^{-2a} \tag{3.109}$$

and

$$\left| \text{Var}^{\mathcal{G}_n}(\psi_{\mathcal{G}_n}(y) | \sigma(A_y)) - \frac{d}{d-1} \right| \le n^{-2a}. \tag{3.110}$$

Proof. We follow the argument of Proposition 3.4.1, with a few adjustments.
First, the bounds in (3.37) and (3.38) are in fact e^{-ca_n} for some constant $c > 0$, and one can replace a_n by r_n.
Second, concerning the proof of (3.39), note that (3.41) still holds for large enough n since for $k \ge 0.3 \log n$ and a constant $\gamma > 0$ that depends neither on k nor on n,

$$\mathbb{P}(Z_k \le r_n) \le \mathbb{P}(Z_k \le 0.05 \log n) \le \mathbb{P}(Z_k \le \tfrac{d-2}{2d} k) \le e^{-\gamma k}.$$

The condition $|A| \leq n^{2/3}$ implies that $\mathbf{E}_{\pi_n}[H_A] \geq n^{1/3}/4$, so that we can adapt the argument below (3.40), as well as the proof of (3.42). $\square$

The second variant of Proposition 3.4.1 corresponds to Case 1 and Case 2 of the first phase of the exploration. Let us introduce some notations before stating the Lemma. Recall that for $m \geq 3$, G_m is the connected (and infinite) d-regular graph with a unique cycle C_m of length m. Recall the definitions of $\alpha_k, \beta_k, \gamma_k, \alpha'_k$ and γ'_k from (3.86) and (3.87).
For $k \geq 0$, let z_k be a vertex at distance k of the cycle C_m in G_m, and $\overline{z_k}$ be a neighbour of z_k at distance $k+1$ of C_m. Note that $B_{G_m}(z_k, \overline{z_k}, r_n)$ (the subgraph of G_m obtained by taking all paths of length r_n starting at z_k and not going through $\overline{z_k}$) contains C_m if and only if $k \leq r_n - \lceil m/2 \rceil$.

Lemma 3.7.8. *If $a > 0$ is small enough, then the following holds for n large enough (uniformly in $m \geq 3$).*
Assume that $\mathcal{G}_n$ is a good graph, that $A \subseteq V_n$ is such that

- $|A| \leq n^{2/3}$,
- *A is a tree, and*
- $\max_{z \in A} |\psi_{\mathcal{G}_n}(z)| \leq \log^{2/3} n$.

Case 1. *Let $y \in V_n$, and suppose that*

- *y has a neighbour $\overline{y}$ in A,*
- *for some $1 \leq k < m$, there exists $\widehat{y}$ in A, a path P of length $m-k$ from y to $\widehat{y}$ whose only vertex in A is $\widehat{y}$, and a path P' in A of length $k-1$ from $\widehat{y}$ to $\overline{y}$, so that $C := P \cup P' \cup (\overline{y}, y)$ is a cycle of length m (and $\widehat{y} = \overline{y}$ if $k = 1$), and*
- $\mathrm{tx}(B_{\mathcal{G}_n}(A \cup C, r_n)) = 1$.

Then

$$\left|\mathbb{E}^{\mathcal{G}_n}[\psi_{\mathcal{G}_n}(y)|\sigma(A)] - \alpha_k \psi_{\mathcal{G}_n}(\overline{y}) - \beta_k \psi_{\mathcal{G}_n}(\widehat{y})\right| \leq n^{-2a} \tag{3.111}$$

and

$$\left|\mathrm{Var}^{\mathcal{G}_n}(\psi_{\mathcal{G}_n}(y)|\sigma(A)) - \gamma_k\ \right| \leq n^{-2a}. \tag{3.112}$$

Case 2. *Let $y \in V_n$, and suppose that*

- *y has a unique neighbour $\overline{y}$ in A,*
- *for some $1 \leq k \leq r_n - \lceil m/2 \rceil$, $B_{\mathcal{G}_n}(y, \overline{y}, r_n)$ is isomorphic to $B_{G_m}(z_k, \overline{z_k}, r_n)$, and*
- $\mathrm{tx}(B_{\mathcal{G}_n}(A \cup P \cup C, r_n)) = 1$, *where C is the cycle in $B_{\mathcal{G}_n}(y, \overline{y}, r_n)$ and P the path from y to C.*

Then
$$\left|\mathbb{E}^{\mathcal{G}_n}[\psi_{\mathcal{G}_n}(y)|\sigma(A)] - \alpha'_k\psi_{\mathcal{G}_n}(\overline{y})\right| \leq n^{-2a} \tag{3.113}$$
and
$$\left|\mathrm{Var}^{\mathcal{G}_n}(\psi_{\mathcal{G}_n}(y)|\sigma(A)) - \gamma'_k\ \right| \leq n^{-2a}. \tag{3.114}$$

Proof. We will only show (3.113) and (3.114). The other proofs are very similar and left to the reader.

We start with the proof of (3.113). We follow the proof scheme of (3.34). Let τ be the hitting time of $\partial B_{\mathcal{G}_n}(y, \overline{y}, r_n) \setminus \{y\}$ by a SRW $(X_k)_{k\geq 0}$. Note that $\{H_A \leq \tau\} \subseteq \{X_{H_A} = y\}$. We write

$$\begin{aligned}\mathbb{E}^{\mathcal{G}_n}[\psi_{\mathcal{G}_n}(y)|\sigma(A)] =& \mathbf{P}_y^{\mathcal{G}_n}(X_{H_A} = \overline{y}, H_A \leq \tau)\psi_{\mathcal{G}_n}(\overline{y}) + \mathbf{E}_y^{\mathcal{G}_n}[\psi_{\mathcal{G}_n}(X_{H_A})\mathbf{1}_{H_A>\tau}] \\ &- \frac{\mathbf{E}_y^{\mathcal{G}_n}[H_A]}{\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[H_A]}\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[\psi_{\mathcal{G}_n}(X_{H_A})].\end{aligned}$$

Since $B_{\mathcal{G}_n}(y, \overline{y}, r_n)$ and $B_{G_m}(z_k, \overline{z_k}, r_n)$ are isomorphic,

$$\mathbf{P}_y^{\mathcal{G}_n}(X_{H_A} = \overline{y}, H_A \leq \tau) = \mathbf{P}_{z_k}^{\mathcal{G}_m}(X_{H_{\partial B_{G_m}(z_k,\overline{z_k},r_n)}} = \overline{z_k}) =: \alpha''_k.$$

Since

$$\{X_{H_{\partial B_{G_m}(z_k,\overline{z_k},r_n)}} = \overline{z_k}\} \subseteq \{H_{\{\overline{z_k}\}} < +\infty\}, \text{ we have } \alpha''_k \leq \alpha'_k.$$

Reciprocally, on $\{H_{\{\overline{z_k}\}} < +\infty\} \setminus \{X_{H_{\partial B_{G_m}(z_k,\overline{z_k},r_n)}} = \overline{z_k}\}$, a SRW starting at distance r_n of C_m has to reach C_m. As in the proof of (3.37), a comparison with a biased SRW on $\mathbb{Z}$ shows that this happens with a probability $O(e^{-cr_n})$ for some constant $c > 0$ uniquely depending on d and we get that if a small enough, then for large enough n, $|\mathbf{P}_y^{\mathcal{G}_n}(X_{H_A} = \overline{y}, H_A \leq \tau) - \alpha'_k| \leq n^{-3a}$. It remains to establish

$$\left|\mathbf{E}_y^{\mathcal{G}_n}[\psi_{\mathcal{G}_n}(X_{H_A})\mathbf{1}_{H_A>\tau}] - \frac{\mathbf{E}_y^{\mathcal{G}_n}[H_A]}{\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[H_A]}\mathbf{E}_{\pi_n}^{\mathcal{G}_n}[\psi_{\mathcal{G}_n}(X_{H_A})]\right| \leq n^{-3a}. \tag{3.115}$$

To do so, one adapts the proofs of (3.38) and (3.39) exactly as in the proof of Lemma 3.7.7: if $H_A > \tau$, $(X_k)_{k\geq 0}$ leaves $B_{\mathcal{G}_n}(A \cup C \cup P, r_n)$ before hitting A. By Lemma 3.4.2 with $s = r_n$, if a is small enough, (X_k) does not hit A within the next $\log^2 n$ steps with probability at least

$$(1 - (d-1)^{-r_n})^{\log^2 n} \geq 1 - 2\log^2 n\ \ (d-1)^{-r_n} \geq 1 - n^{-4a}.$$

Then, we use Corollary 2.1.5 of [126] as six lines above (3.40): after $\lfloor \log^2 n \rfloor$ steps, the fact that $\mathcal{G}_n$ is an expander forces the empirical distribution of X_k to be very close to the uniform distribution π_n. (3.113) follows.

For (3.114), we follow the proof scheme of (3.35). If τ' is the exit time of $B_{\mathcal{G}_n}(y, \overline{y}, r_n)$, we have

$$\begin{aligned}\left|\mathrm{Var}^{\mathcal{G}_n}(\psi_{\mathcal{G}_n}(y)|\sigma(A)) - \gamma'_k\right| \leq& \left|G_{\mathcal{G}_n}(y,y) - \mathbf{E}_y^{\mathcal{G}_n}\left[G_{\mathcal{G}_n}\left(y, X_{H_A}\right)\mathbf{1}_{H_A=\tau'}\right] - \gamma'_k\right| \\ &+ \left|\mathbf{E}_y^{\mathcal{G}_n}\left[G_{\mathcal{G}_n}\left(y, X_{H_A}\right)\mathbf{1}_{H_A>\tau'}\right] - \frac{\mathbf{E}_y^{\mathcal{G}_n}[H_A]}{\mathbf{E}_{\pi_n}[H_A]}\mathbf{E}_{\pi_n}^{\mathcal{G}_n}\left[G_{\mathcal{G}_n}\left(y, X_{H_A}\right)\right]\right|.\end{aligned} \tag{3.116}$$

We deal with the second term of the RHS as (3.115) to show that it is $O(n^{-3a})$. As below (3.45), we have

$$\mathbf{E}_y^{\mathcal{G}_n}\left[G_{\mathcal{G}_n}\left(y, X_{H_A}\right)\mathbf{1}_{H_A=\tau'}\right] = \mathbf{E}_y^{\mathcal{G}_n}\left[G_{\mathcal{G}_n}\left(y, X_{\tau'}\right)\right] - O(n^{-3a})$$

if a is small enough, by (3.11). Now, by (3.17) applied to $D := B_{\mathcal{G}_n}(y, \overline{y}, r_n)$ (note that $T_D = \tau'$) and (3.41) (which still holds, as remarked in the proof of Lemma 3.7.7), we get

$$|G_{\mathcal{G}_n}(y,y) - \mathbf{E}_y^{\mathcal{G}_n}\left[G_{\mathcal{G}_n}\left(y, X_{\tau'}\right)\right] - G_{\mathcal{G}_n}^D(y,y)| \leq \frac{\mathbf{E}_y^{\mathcal{G}_n}[\tau']}{n} = O(n^{-3a}).$$

But D is isomorphic to $E := B_{G_m}(z_k, \overline{z_k}, r_n)$, so that

$$G_{\mathcal{G}_n}^D(y,y) = G_{G_m}^E(z_k,z_k) = G_{G_m}(z_k,z_k) - \mathbf{P}_{z_k}^{G_m}(T_E = \overline{z_k})G_{G_m}(z_k,\overline{z_k}) - \mathbf{P}_{z_k}^{G_m}(T_B \neq \overline{z_k})G_{G_m}(z_k,z)$$

for any $z \in \partial B_{G_m}(E,1) \setminus \{\overline{z_k}\}$, by cylindrical symmetry of $B_{G_m}(E,1)$. One checks easily that if a is small enough, then for n large enough,

$$G_{\mathcal{G}_n}^D(y,y) = G_{G_m}^E(z_k,z_k) = G_{G_m}(z_k,z_k) - \mathbf{P}_{z_k}^{G_m}(T_E = \overline{z_k})G_{G_m}(z_k,\overline{z_k}) + O(n^{-3a}).$$

One easily adapts the reasoning leading to (3.37), despite the presence of one cycle, to get

$$\mathbf{P}_{z_k}^{G_m}(T_E = \overline{z_k}) = \mathbf{P}_{z_k}^{G_m}(H_{\{\overline{z_k}\}} < +\infty) + O(n^{-3a}) \text{ for } a \text{ small enough.}$$

Note indeed that $\{T_E = \overline{z_k}\} \subseteq \{H_{\{\overline{z_k}\}} < +\infty\}$. Reciprocally, if $z \in \partial B_{G_m}(E,1) \setminus \{\overline{z_k}\}$, a SRW starting at z has a probability decaying exponentially with r_n to reach $\overline{z_k}$, since there are at most two injective paths from z to $\overline{z_k}$, and each contains at least $r_n - 3$ vertices where the SRW has a positive probability (only depending on d) to enter a subtree isomorphic $\mathbb{T}_d^+$ and to never leave it.
Since $\gamma_k' = G_{G_m}(z_k,z_k) - \mathbf{P}_{z_k}^{G_m}(H_{\{\overline{z_k}\}} < +\infty)G_{G_m}(z_k,\overline{z_k})$, we obtain

$$|G_{\mathcal{G}_n}^D(y,y) - \gamma_k'| = O(n^{-3a}).$$

All in all, we get that the first term of the RHS of (3.116) is $O(n^{-3a})$, and (3.114) follows. □

Proof of Proposition 3.7.6. We proceed as below (3.54) in the proof of Proposition 3.5.1. Denote

$$\mathcal{E}_n := \{\exists y \in R_1^\psi, |\psi_{\mathcal{G}_n}(y) - \widehat{\psi_{\mathcal{G}_n}}(y)| \geq n^{-a}\} \cap \mathcal{E}_{7,n} \cap \mathcal{S}_1(x).$$

On $\mathcal{E}_n$, $\widehat{\psi_{\mathcal{G}_n}}$ is defined on at most $K_0 \log n$ vertices by D2 (and our choice of $K' = K_0$), so that by the triangle inequality, either $|\widehat{\psi_{\mathcal{G}_n}}(x) - \psi_{\mathcal{G}_n}(x)| \geq n^{-a}\log^{-2} n$, or there exists y such that

$$|\widehat{\psi_{\mathcal{G}_n}}(y) - \psi_{\mathcal{G}_n}(y)| \geq n^{-a}\log^{-2} n + \sup_{\overline{y} \in R_y^\psi} |\widehat{\psi_{\mathcal{G}_n}}(\overline{y}) - \psi_{\mathcal{G}_n}(\overline{y})|,$$

where R_y^ψ is the set of vertices where $\widehat{\psi_{\mathcal{G}_n}}$ has been defined before $\widehat{\psi_{\mathcal{G}_n}}(y)$. Let

$$\mathcal{E}(y) := \{|\widehat{\psi_{\mathcal{G}_n}}(y) - \psi_{\mathcal{G}_n}(y)| \geq n^{-a}\log^{-2} n + \sup_{\overline{y} \in R_y^\psi} |\widehat{\psi_{\mathcal{G}_n}}(\overline{y}) - \psi_{\mathcal{G}_n}(\overline{y})|\} \cap \mathcal{E}_n$$

For $y \neq x$, we can apply Lemma 3.7.8 (if $\widehat{\psi_{\mathcal{G}_n}}(y)$ is defined in Case 1 or Case 2) or Lemma 3.7.7 (in the other cases), so that

$$\mathbb{P}_{ann}(\mathcal{E}(y)) \leq \mathbb{P}_{ann}(n^{-2a}|\xi_y| \geq n^{-a}\log^{-2} n - n^{-2a}) \leq \mathbb{P}_{ann}(|\xi_y| \geq n^{a/2}) \leq n^{-4}$$

by the exponential Markov inequality, and $\mathbb{P}_{ann}(\mathcal{E}(x)) \leq n^{-4}$ by the same argument, where we set $\mathcal{E}(x) := \{|\widehat{\psi_{\mathcal{G}_n}}(x) - \psi_{\mathcal{G}_n}(x)| \geq n^{-a}\log^{-2} n\} \cap \mathcal{E}_n$. Hence, we get by a union bound on $y \in R_1^\psi$ that for large enough n,

$$\mathbb{P}_{ann}(\mathcal{E}_n) \leq n^{-4}K_0 \log n \leq n^{-3}.$$

The conclusion follows. □

The field $\psi_{\mathcal{G}_n}$ on the second phase: proof of (3.106)

It is enough to show that for each $y_{q,i}$ whose exploration in the second phase is successful,

$$\mathbb{P}_{ann}\left(\{\sup_{z\in T_{y_{q,i}}\setminus\{y_{q,i}\}} |\widehat{\psi_{\mathcal{G}_n}}(z) - \psi_{\mathcal{G}_n}(z)| \geq \tfrac{\log^{-1} n}{2}\} \cap \mathcal{E}_{7,n}\right) \leq n^{-4},$$

where $T_{y_{q,i}}$ is its exploration tree, and to conclude by a union bound on $y_{q,i}$. This follows from a straightforward adaptation of the reasoning below (3.54). Note that the n^{-3} in the RHS of (3.55) can be replaced by any polynomial in n.

The field $\psi_{\mathcal{G}_n}$ on the third phase: proof of (3.107)

By symmetry, it is enough to consider the case $i = 1$. Following readily the argument of the proof of (3.70), we get that the probability that $E^{\geq h}_{\psi_{\mathcal{G}_n}}$ percolates through a given J-joining ball is at least $\log^{\gamma(K_8/3-1)} n$, for any J. By (3.96) and (3.97), and a union bound on every couple $(J, q) \in \{1, \dots, s_0\} \times \{1, \dots m\}$,

$$\mathbb{P}_{ann}((\mathcal{S}_2(x) \setminus \mathcal{S}^\psi_{3,1}(x)) \cap \mathcal{E}_{7,n}) \leq 2n^{-4} + s_0 m \mathbb{P}(Z = 0),$$

where $Z \sim \mathrm{Bin}(\lfloor \log^{\gamma-3\kappa-18} n\rfloor, \log^{\gamma(K_8/3-1)} n)$. If κ and γ/κ are large enough, then for n large enough,

$$\mathbb{P}(Z = 0) = (1 - \log^{\gamma(K_8/3-1)} n)^{\lfloor \log^{\gamma-3\kappa-18} n\rfloor} \leq n^{-4},$$

and (3.107) follows.

3.8 Properties of $\mathcal{C}_1^{(n)}$

3.8.1 The local limit: proof of Theorem 3.1.3

Proof of Theorem 3.1.3. The proof mimics the reasoning of Lemma 3.6.3. Let $k \geq 0$ and let T be a rooted tree of height k, with no vertex of degree more than d. Let $x \in V_n$. We perform an exploration as in Section 3.5.1. Denote $\mathcal{S}_T(x)$ the event that the exploration is successful, that $\mathcal{S}_T(x) \subseteq \mathcal{C}_x^{\mathcal{G}_n,h}$ and that $B_{\mathcal{G}_n}(x,k) = T$. We claim that

$$\mathbb{P}_{ann}(\mathcal{S}_T(x)) \underset{n\to+\infty}{\longrightarrow} \mathbb{P}^{\mathbb{T}_d}(B_{\mathbb{T}_d}(\circ,k) = T, |\mathcal{C}_\circ^h| = +\infty). \tag{3.117}$$

The proof goes as those of Lemma 3.5.3 and Proposition 3.5.1. In the proof of Lemma 3.5.3, replace $\mathcal{F}_j^{(n)}$ by $\mathcal{F}_{T,j}^{(n)} := \mathcal{F}_j^{(n)} \cap \{B_{\mathcal{C}_\circ^h}(\circ,k) = T\}$ and $\mathcal{F'}_j^{(n)}$ by $\mathcal{F'}_{T,j}^{(n)} := \mathcal{F'}_j^{(n)} \cup \{B_{\mathcal{C}_\circ^h}(\circ,k) \neq T\}$, for every $j \geq 1$. We also check easily that

$$\mathbb{P}^{\mathbb{T}_d}(\mathcal{C}_\circ^{h+\log^{-1} n} \cap B_{\mathbb{T}_d}(\circ,k) = \mathcal{C}_\circ^{h-\log^{-1} n} \cap B_{\mathbb{T}_d}(\circ,k)) \to 1$$

in order to determine $B_{\mathcal{G}_n}(x,k)$ $\mathbb{P}_{ann}$-w.h.p., as Proposition 3.5.1 only ensures that $\mathcal{C}_\circ^{h+\log^{-1} n} \cap B_{\mathbb{T}_d}(\circ,k) \subseteq B_{\mathcal{G}_n}(x,k)$ w.h.p.
Moreover, we get as for (3.77):

$$\sup_{x,y\in V_n} |\mathrm{Cov}_{ann}(\mathcal{S}_T(x), \mathcal{S}_T(y))| \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.118}$$

Let $\varepsilon > 0$. Applying Bienaymé-Chebyshev's inequality as in Lemma 3.6.3, we get

$$\mathbb{P}_{ann}(|\,|\mathcal{S}_T| - \mathbb{P}^{\mathbb{T}_d}(B_{\mathbb{T}_d}(\circ,k) = T, |\mathcal{C}_\circ^h| = +\infty)n| \leq \varepsilon n) \underset{n\to+\infty}{\longrightarrow} 1, \tag{3.119}$$

where $\mathcal{S}_T$ is the set of vertices $x \in V_n$ such that $\mathcal{S}_T(x)$ holds. Let $\mathcal{S} \subseteq V_n$ be the set of vertices such that their exploration is successful. By Theorem 3.1.1 and a reasoning as in the proof of Lemma 3.6.3, with $\mathbb{P}_{ann}$-probability $1 - o(1)$,

(i) $|\mathcal{C}_2^{(n)}| \leq n^{1/3}$, so that $\mathcal{S}_T \subseteq \mathcal{S} \cap \mathcal{C}_1^{(n)}$,

(ii) $|\,|\mathcal{C}_1^{(n)}| - \eta(h)n| \leq \varepsilon n$,

(iii) $|\,|\mathcal{S}| - \eta(h)n| \leq \varepsilon n$.

Suppose that these three assumptions and (3.119) hold. By (i), $\mathcal{S}_T = \mathcal{S} \cap V_n^{(T)}$. Therefore, $|\mathcal{S}_T| \leq |V_n^{(T)}| \leq |\mathcal{S}_T| + |\mathcal{C}_1^{(n)} \cap \mathcal{S}^c|$, so that by (ii), (iii) and (3.119),

$$(\mathbb{P}^{\mathbb{T}_d}(B_{\mathbb{T}_d}(\circ,k) = T, |\mathcal{C}_\circ^h| = +\infty) - \varepsilon)n \leq |V_n^{(T)}| \leq (\mathbb{P}^{\mathbb{T}_d}(B_{\mathbb{T}_d}(\circ,k) = T, |\mathcal{C}_\circ^h| = +\infty) + 3\varepsilon)n.$$

Moreover, $|\,|\mathcal{C}_1^{(n)}| - \eta(h)n| \leq \varepsilon n$ by (ii), so that for ε small enough,

$$\frac{\mathbb{P}^{\mathbb{T}_d}(B_{\mathbb{T}_d}(\circ,k) = T, |\mathcal{C}_\circ^h| = +\infty)}{\eta(h)} - \sqrt{\varepsilon} \leq \frac{|V_n^{(T)}|}{|\mathcal{C}_1^{(n)}|} \leq \frac{\mathbb{P}^{\mathbb{T}_d}(B_{\mathbb{T}_d}(\circ,k) = T, |\mathcal{C}_\circ^h| = +\infty)}{\eta(h)} + \sqrt{\varepsilon}.$$

But

$$\begin{aligned}\frac{\mathbb{P}^{\mathbb{T}_d}(B_{\mathbb{T}_d}(\circ,k) = T, |\mathcal{C}_\circ^h| = +\infty)}{\eta(h)} &= \frac{\mathbb{P}^{\mathbb{T}_d}(B_{\mathbb{T}_d}(\circ,k) = T, |\mathcal{C}_\circ^h| = +\infty)}{\mathbb{P}^{\mathbb{T}_d}(|\mathcal{C}_\circ^h| = +\infty)} \\ &= \mathbb{P}^{\mathbb{T}_d}(B_{\mathbb{T}_d}(\circ,k) = T \mid |\mathcal{C}_\circ^h| = +\infty).\end{aligned}$$

And since we can take ε arbitrarily small, the conclusion follows.

□

3.8.2 The core and the kernel: proof of (3.4) and (3.5)

We now prove (3.4) and (3.5) of Theorem 3.1.2, starting with (3.4). Let K_1 (resp. K_2) be the probability that $\circ$ has at least 2 (resp. 3) children with an infinite offspring in $\mathcal{C}_\circ^h$, under $\mathbb{P}^{\mathbb{T}_d}$. Suppose that for $x, y \in V_n$,

$$\mathbb{P}_{ann}(x \in \mathbf{C}^{(n)}) \underset{n\to+\infty}{\longrightarrow} K_1 \tag{3.120}$$

and

$$\mathrm{Cov}_{ann}(\mathbf{1}_{x\in\mathbf{C}^{(n)}}, \mathbf{1}_{y\in\mathbf{C}^{(n)}}) \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.121}$$

Then (3.4) follows by a second moment argument as in Lemma 3.6.3. Hence, it is enough to prove (3.120) and (3.121).

Proof of (3.120). For $x \in V_n$, we perform the exploration in Section 3.5.1 from x, replacing C2 by the following condition: for every neighbour v of x, stop exploring the subtree from v at step $k+1$ if the k-offspring of v has at least $n^{1/2}b_n$ vertices. Stop the exploration if this happens for at least two neighbours of x. In this case, say that the exploration is successful.
We adapt easily the proofs of Lemma 3.5.3 and Proposition 3.5.1 to show that for $x \in V_n$,

$$\mathbb{P}_{ann}(\text{the exploration from } x \text{ is successful}) \underset{n\to+\infty}{\longrightarrow} K_1. \tag{3.122}$$

Indeed, K_1 is the probability that the realization of $\varphi_{\mathbb{T}_d}$ to which we couple $\psi_{\mathcal{G}_n}$ is such that $\circ$ has at least two children with an infinite offspring. Then, as in Lemma 3.5.3, there is a probability $1-o(1)$ that the offspring of these children grows at an exponential rate close to λ_h (Proposition 3.3.8). Thus, letting $\mathcal{F}_{v,k}^{(n)} := \cup_{1\le j\le k}\{\text{the } j\text{-offspring of } v \text{ has at least } n^{1/2}b_n \text{ vertices}\}$ for every child v of $\circ$, and $\mathcal{F}_{\text{core},k}^{(n)} := \cup_{v_1,v_2 \text{ children of } \circ}(\mathcal{F}_{v_1,k-1}^{(n)} \cap \mathcal{F}_{v_2,k-1}^{(n)})$, we get that

$$\mathbb{P}_{ann}(\cup_{1\le k\le \lfloor \log_{\lambda_h} n\rfloor}\mathcal{F}_{\text{core},k}^{(n)}) \underset{n\to+\infty}{\longrightarrow} K_1$$

and

$$\mathbb{P}_{ann}(\cup_{1\le k\le \lfloor \log_{\lambda_h} n\rfloor}\{\text{at most one child of } \circ \text{ has a non-empty } k\text{-offspring}\}) \underset{n\to+\infty}{\longrightarrow} 1-K_1.$$

As for (3.53), we see that we do not meet any cycle with $\mathbb{P}_{ann}$-probability $1-o(1)$. This yields (3.122).
If the exploration from x is successful, let x_1, x_2 be two children of x such that the exploration subtrees T_{x_1} and T_{x_2} from x_1 and x_2 satisfy $\min(|\partial T_{x_1}|, |\partial T_{x_2}|) \ge n^{1/2}b_n$. Then, let $K > 0$ and let $w_1, \dots, w_{\lfloor K\log n\rfloor} \in V_n$ be vertices that have not been met in the exploration from x. Proceed to their w-exploration as described in the first part of the construction in Section 3.7.4.
By Remark 3.5.2, $o(\sqrt{n})$ vertices are seen during the exploration from x and the w-explorations. Applying (3.14) with $k=1$, $m_0, m_1, m_E, m = o(\sqrt{n})$, we get that with $\mathbb{P}_{ann}$-probability $1-o(1)$, none of the w-explorations intersects the exploration from x, and no w_i is spoiled or back-spoiled. As in (3.104), we get that for each w_i, its w-exploration has probability at least $\eta(h)/2$ to be

w-successful. Hence with $\mathbb{P}_{ann}$-probability at least $1-(1-\eta(h)/2)^{\lfloor K\log n\rfloor} = 1-o(1)$, there exists i_0 such that the w-exploration from w_{i_0} is successful. Denote $\partial T_{w_{i_0}}$ the boundary of its exploration tree. Write

$$\begin{aligned}\mathcal{S}_{\text{core}}(x) :=&\{\text{the exploration from } x \text{ is successful}\}\cap\\ &\{\exists i_0 \ \geq\ 1,\ \text{ the } w\text{-exploration from } w_{i_0} \text{ is successful}\}.\end{aligned}$$

We have just shown that

$$\liminf_{n\to+\infty}\mathbb{P}_{ann}(\mathcal{S}_{\text{core}}(x)) \geq \liminf_{n\to+\infty}\mathbb{P}_{ann}(\text{the exploration from } x \text{ is successful}) \geq K_1. \tag{3.123}$$

Next, we grow joining balls from ∂T_{x_1} to $\partial T_{w_{i_0}}$, and then from ∂T_{x_2} to $\partial T_{w_{i_0}}$. We proceed as in the second part of the construction in Section 3.7.4 with $m=2$ and $s_0=1$. Similarly to (3.96), we get that

$$\mathbb{P}_{ann}(\mathcal{S}_{\text{core}}(x)\cap(\mathcal{S}_{\text{core,joining,1}}\cup\mathcal{S}_{\text{core,joining,2}})) = o(1),$$

with $\mathcal{S}_{\text{core,joining,i}} := \{\text{there are less than } \log^{\gamma-3\kappa-18} n \text{ joining balls from } \partial T_{x_i} \text{ to } \partial T_{w_{i_0}}\}$ for $i = 1,2$.

Finally, we reveal $\psi_{\mathcal{G}_n}$ on the exploration from x, on the w-exploration from w_{i_0}, on the joining balls from ∂T_{x_1} to $\partial T_{w_{i_0}}$, and on the joining balls from ∂T_{x_2} to $\partial T_{w_{i_0}}$, in that order. Denote $\mathcal{S}_{\text{core,connect}}$ the event that there exists a joining ball B_1 from ∂T_{x_1} to $\partial T_{w_{i_0}}$ and another B_2 from ∂T_{x_2} to $\partial T_{w_{i_0}}$ such that $\min_{y\in T_{x_1}\cup T_{x_2}\cup T_{w_{i_0}}\cup B_1\cup B_2}\psi_{\mathcal{G}_n}(y)\geq h$. As in the proof of Proposition 3.6.2, we get that $\mathbb{P}_{ann}(\mathcal{S}_{\text{core}}(x)\cap\mathcal{S}^c_{\text{core,connect}}) = o(1)$. Hence

$$\begin{aligned}\liminf\nolimits_{n\to+\infty}\mathbb{P}_{ann}(&x,x_1,x_2,z \text{ are in a cycle of } \mathcal{C}_x^{\mathcal{G}_n} \text{ and } |\mathcal{C}_x^{\mathcal{G}_n}|\geq n^{1/3})\\ &\geq \liminf\nolimits_{n\to+\infty}\mathbb{P}_{ann}(\mathcal{S}_{\text{core}}(x)).\end{aligned}$$

If $|\mathcal{C}_2^{(n)}| < n^{1/3}$, this cycle is in $\mathcal{C}_1^{(n)}$, and thus in $\mathbf{C}^{(n)}$ so that by (3.3), for large enough n and any $x\in V_n$, one has by (3.123):

$$\liminf\nolimits_{n\to+\infty}\mathbb{P}_{ann}(x\in\mathbf{C}^{(n)}) \geq \liminf\nolimits_{n\to+\infty}\mathbb{P}_{ann}(\mathcal{S}_{\text{core}}(x)) \geq K_1\ .$$

Reciprocally, for $x\in V_n$, turn the exploration into a lower exploration, replacing $h+\log^{-1} n$ by $h-\log^{-1} n$ (as in Section 3.5.2). Say that the lower exploration from x is aborted if for some $k\leq\log\log n$, at most one child of x has a non-empty $(k-1)$-offspring. Let $\mathcal{A}_{\text{core}}(x) := \{\text{the lower exploration from } x \text{ is aborted}\}$. We get as in the proof of (3.122):

$$\mathbb{P}_{ann}(\mathcal{A}_{\text{core}}(x)) \underset{n\to+\infty}{\longrightarrow} 1-K_1. \tag{3.124}$$

Moreover, revealing $\psi_{\mathcal{G}_n}$ on T_x, we can apply Proposition 3.4.1 as below (3.54) to get that

$$\mathbb{P}_{ann}(\mathcal{A}_{\text{core}}(x)\cap\{\mathcal{C}_x^{\mathcal{G}_n,h}\cap B_{\mathcal{G}_n}(x,\lfloor\log\log n\rfloor)\subseteq T_x\}) \underset{n\to+\infty}{\longrightarrow} 1-K_1. \tag{3.125}$$

For each neighbour y of x, denote $\mathcal{C}_y$ its connected component in $\mathcal{C}_x^{\mathcal{G}_n} \setminus \{x\}$. If the exploration is aborted and $\mathcal{C}_x^{\mathcal{G}_n,h} \cap B_{\mathcal{G}_n}(x, \lfloor \log\log n \rfloor) \subseteq T_x$, then x has at most one neighbour y such that $\mathcal{C}_y \cup \{x\}$ is not a tree. Hence $x \notin \mathbf{C}^{(n)}$. Thus, $\liminf_{n\to+\infty} \mathbb{P}_{ann}(x \notin \mathbf{C}^{(n)}) \geq 1 - K_1$ and (3.120) follows. □

Proof of (3.121). By (3.120), for $x, y \in V_n$,

$$\mathbb{P}_{ann}(x \in \mathbf{C}^{(n)})\mathbb{P}_{ann}(y \in \mathbf{C}^{(n)}) \underset{n\to+\infty}{\longrightarrow} K_1^2.$$

It remains to show that $\mathbb{P}_{ann}(x, y \in \mathbf{C}^{(n)}) \underset{n\to+\infty}{\longrightarrow} K_1^2$.
Perform the exploration from x as in the beginning of the proof of (3.120), then do the same from y (and stop the latter if it reaches a vertex of the exploration from x). Since $o(\sqrt{n})$ vertices are revealed during these explorations (see Remark 3.5.2), then by (3.14), the probability that the exploration from y meets that of x is $o(1)$. Thus by (3.122),

$$\mathbb{P}_{ann}(\text{the explorations from } x \text{ and } y \text{ are both successful}) \underset{n\to+\infty}{\longrightarrow} K_1^2.$$

Then, let x_1, x_2 (resp. y_1, y_2) be the children of x (resp. y) whose exploration is successful. We complete the exploration in a fashion similar to that above (3.123). Let $K > 0$ and let $w_1, \ldots, w_{\lfloor K \log n \rfloor} \in V_n$ be vertices that have not been met in the explorations from x and y, and proceed to their w-exploration. If there exists $i_0 \geq 1$ such that the w-exploration from w_{i_0} is successful, build joining balls from $\partial T_{x_1}, \partial T_{x_2}, \partial T_{y_1}$ and ∂T_{y_2} to $T_{w_{i_0}}$. Finally, reveal $\psi_{\mathcal{G}_n}$ on T_x, on T_y, on $T_{w_{i_0}}$ and on the joining balls from $\partial T_{x_1}, \partial T_{x_2}, \partial T_{y_1}$ and ∂T_{y_2}, in that order. As in the proof of (3.123) and below, we get that

$$\liminf_{n\to+\infty} \mathbb{P}_{ann}(x, y \in \mathbf{C}^{(n)}) \geq K_1^2. \tag{3.126}$$

Conversely, if we perform the lower explorations from x and y as defined in the end of the proof of (3.120), we easily get that

$$\mathbb{P}_{ann}(\exists z \in \{x, y\}, \text{ the lower exploration from } z \text{ is aborted}) \underset{n\to+\infty}{\longrightarrow} 1 - K_1^2.$$

Then, we reveal $\psi_{\mathcal{G}_n}$ on T_z. Following the reasoning below (3.54), we get that $\mathbb{P}_{ann}$-w.h.p., $B_{\mathcal{C}_z^{\mathcal{G}_n,h}}(z, \lfloor \log\log n \rfloor) \subseteq T_z$, and thus

$$\mathbb{P}_{ann}(\{\text{the lower exploration from } z \text{ is aborted}\} \cap \{z \in \mathbf{C}^{(n)}\}) = o(1).$$

This yields

$$\liminf_{n\to+\infty} \mathbb{P}_{ann}(\exists z \in \{x, y\}, z \notin \mathbf{C}^{(n)}) \geq 1 - K_1^2.$$

Together with (3.126), this concludes the proof. □

This reasoning can be readily adapted to prove (3.5), with a modification of the exploration (requiring that at least three children of x have a successful exploration).

3.8.3 The typical distance: proof of (3.7)

The proof of (3.7) goes as that of (3.2), with a slight modification of the explorations of Section 3.5. Those explorations were indeed stopped after at most $\log_{\lambda_h} n$ steps. But since around $\sqrt{n}$ vertices were explored, and since the growth rate of $\mathcal{C}_\circ^h$ is close to λ_h, we can expect that a successful exploration lasts in fact $(1/2 + o(1)) \log_{\lambda_h} n$ steps. Then, connecting two such explorations as in Proposition 2.5.16 (adding an additional distance $\Theta(\log\log n) = o(\log n)$) yields the diameter (since reciprocally, explorations lasting less steps will be too small to be connected).

Proof of (3.7). Fix $\varepsilon \in (0,1)$. We start by the upper bound. In the exploration of Section 3.5.1, replace C4 by the condition: stop the exploration if $k \geq (1/2+\varepsilon/3)\log_{\lambda_h} n$. By Proposition 3.3.8, as $n \to +\infty$:

$$\mathbb{P}^{\mathbb{T}_d}\left(|\mathcal{Z}^h_{\lfloor (1/2+\varepsilon/3)\log_{\lambda_h} n \rfloor}| > n^{1/2} b_n \,\middle|\, |\mathcal{C}_\circ^{h+\log^{-1} n}| = +\infty \right) \to 1.$$

Thus, Propositions 3.5.1, 3.6.2 and Lemma 3.6.4 remain unchanged. Note that if x, y are connected in $E^{\geq h}_{\psi_{\mathcal{G}_n}}$ via the successful explorations from x and y and the joining balls from ∂T_x to ∂T_y, then for n large enough:

$$d_{E^{\geq h}_{\psi_{\mathcal{G}_n}}}(x,y) \leq 2(1/2+\varepsilon/3)\log_{\lambda_h} n + a'_n \leq (1+\varepsilon)\log_{\lambda_h} n.$$

Then, Lemma 3.6.4 implies that

$$\mathbb{P}_{ann}(\mathcal{E}_{1,n} \cap \mathcal{E}_{2,n} \cap \mathcal{E}_{3,n}) \underset{n\to+\infty}{\longrightarrow} 1,$$

where $\mathcal{E}_{1,n} := \{|\{(x,y) \in V_n^2, d_{E^{\geq h}_{\psi_{\mathcal{G}_n}}}(x,y) \leq (1+\varepsilon)\log_{\lambda_h} n\}| \geq (\eta(h)^2 - \varepsilon)n^2\}$.

We have to check that only $o(n^2)$ of the couples (x,y) described in $\mathcal{E}_{1,n}$ are not in $\mathcal{C}_1^{(n)}$, and that $|\mathcal{C}_1^{(n)}|/n$ is indeed close to $\eta(h)$. Note that by (3.2) and (3.3),

$$\mathbb{P}_{ann}(\mathcal{E}_{1,n} \cap \mathcal{E}_{2,n} \cap \mathcal{E}_{3,n}) \underset{n\to+\infty}{\longrightarrow} 1,$$

where $\mathcal{E}_{1,n} := \{|\{(x,y) \in V_n^2, d_{E^{\geq h}_{\psi_{\mathcal{G}_n}}}(x,y) \leq (1+\varepsilon)\log_{\lambda_h} n\}| \geq (\eta(h)^2 - \varepsilon)n^2\}$,

$$\mathcal{E}_{2,n} := \{\forall i \geq 2,\ |\mathcal{C}_i^{(n)}| \leq K_0 \log n\} \text{ and } \mathcal{E}_{3,n} := \{(\eta(h)-\varepsilon)n \leq |\mathcal{C}_1^{(n)}| \leq (\eta(h)+\varepsilon)n\}.$$

On $\mathcal{E}_{2,n}$, we have $|\{(x,y) \in V_n^2 \setminus (\mathcal{C}_1^{(n)})^2,\ d_{E^{\geq h}_{\psi_{\mathcal{G}_n}}}(x,y) \leq (1+\varepsilon)\log_{\lambda_h} n\}| \leq n^{3/2}$, so that on $\mathcal{E}_{1,n} \cap \mathcal{E}_{2,n}$,

$$|\{(x,y) \in (\mathcal{C}_1^{(n)})^2, d_{\mathcal{C}_1^{(n)}}(x,y) \leq (1+\varepsilon)\log_{\lambda_h} n\}| \geq (\eta(h)^2 - 2\varepsilon)n^2.$$

Thus, on $\mathcal{E}_{1,n} \cap \mathcal{E}_{2,n} \cap \mathcal{E}_{3,n}$:

$$\begin{aligned}\pi_{2,n}\left(\{(x,y) \in (\mathcal{C}_1^{(n)})^2, d_{\mathcal{C}_1^{(n)}}(x,y) \leq (1+\varepsilon)\log_{\lambda_h} n\}\right) &\geq \frac{\eta(h)^2 - 2\varepsilon}{(\eta(h)+\varepsilon)^2} \geq 1 - \frac{2\varepsilon + 2\eta(h)\varepsilon + \varepsilon^2}{(\eta(h)+\varepsilon)^2} \\ &\geq 1 - \frac{(3+2\eta(h))}{\eta(h)^2}\varepsilon.\end{aligned}$$

It remains to show that the typical distance in $\mathcal{C}_1^{(n)}$ is at least $(1-\varepsilon)\log_{\lambda_h} n$. Modify the lower exploration of Section 3.5.2: say that it is aborted if

- C1 did not happen, and
- it is stopped at some step $k \leq (1/2-\varepsilon/2)\log_{\lambda_h} n$, or less than $n^{1/2-\varepsilon/10}$ vertices and half-edges have been seen at step $\lfloor (1/2-\varepsilon/2)\log_{\lambda_h} n \rfloor$.

Suppose that

$$\mathbb{P}_{ann}(\text{the lower exploration from } x \text{ is aborted}) = 1 - o(1). \tag{3.127}$$

For $x, y \in V_n$, perform the lower exploration from x, then that from y, and stop it if it meets a vertex of the exploration from x. This happens with $\mathbb{P}_{ann}$-probability $o(1)$ by (3.14) with $k=1$, $m_0, m_1, m, m_E = o(\sqrt{n})$ (recall Remark 3.5.2).Hence by (3.127):

$$\mathbb{P}_{ann}(\mathcal{E}_{\mathrm{ab}}(x,y)){=}1-o(1), \text{ with } \mathcal{E}_{\mathrm{ab}}(x,y){:=}\{\text{the lower explorations from } x \text{ and } y \text{ are aborted}\},$$

Then, reveal $\psi_{\mathcal{G}_n}$ on the exploration trees T_x and T_y. Applying Proposition 3.4.1 as below (3.54), we get that:

$$\mathbb{P}_{ann}(\mathcal{E}_{\mathrm{ab}}(x,y) \cap \mathcal{E}_x \cap \mathcal{E}_y) = 1 - o(1),$$

with $\mathcal{E}_x{:=}\{B_{\mathcal{C}_x^{\mathcal{G}_n,h}}(x, \lfloor (1/2{-}\varepsilon/2)\log_{\lambda_h} n\rfloor){\subseteq}\, T_x\}$ and $\mathcal{E}_y{:=}\{B_{\mathcal{C}_y^{\mathcal{G}_n,h}}(y, \lfloor (1/2{-}\varepsilon/2)\log_{\lambda_h} n\rfloor){\subseteq}\, T_y\}$.

On $\mathcal{E}_{\mathrm{ab}}(x,y) \cap \mathcal{E}_x \cap \mathcal{E}_y$, if $x, y \in \mathcal{C}_1^{(n)}$, then $d_{\mathcal{C}_1^{(n)}}(x,y) \geq (1-\varepsilon)\log_{\lambda_h} n$. Therefore, for every $x, y \in V_n$, $\mathbb{P}_{ann}(\mathcal{E}_{x,y}) = o(1)$, with

$$\mathcal{E}_{x,y} := \{x, y \in \mathcal{C}_1^{(n)}, d_{\mathcal{C}_1^{(n)}}(x,y) < (1-\varepsilon)\log_{\lambda_h} n\}.$$

Similarly, for all distinct $x, y, z, t \in V_n$, we get that $\mathbb{P}_{ann}(\mathcal{E}_{x,y} \cap \mathcal{E}_{z,t}) = o(1)$, so that

$$\mathrm{Cov}_{ann}(\mathbf{1}_{\mathcal{E}_{x,y}}, \mathbf{1}_{\mathcal{E}_{z,t}}) = o(1).$$

Thus by Bienaymé-Chebyshev's inequality,

$$\mathbb{P}_{ann}(\mathcal{E}_{3,n} \cap \{|\{(x,y) \in V_n^2,\, d_{\mathcal{C}_1^{(n)}}(x,y) \leq (1-\varepsilon)\log_{\lambda_h} n\}| \geq \varepsilon n^2\}) \underset{n\to+\infty}{\longrightarrow} 1.$$

For $\varepsilon > 0$ small enough and n large enough, on

$$\mathcal{E}_{3,n} \cap \{|\{(x,y) \in V_n^2,\, d_{\mathcal{C}_1^{(n)}}(x,y) \leq (1-\varepsilon)\log_{\lambda_h} n\}| \geq \varepsilon n^2\},$$

$\pi_{2,n}\left(\{(x,y) \in (\mathcal{C}_1^{(n)})^2, d_{\mathcal{C}_1^{(n)}}(x,y) \leq (1-\varepsilon)\log_{\lambda_h} n\}\right) \leq \frac{2\varepsilon}{\eta(h)^2}$. This concludes the proof of (3.7).

Thus, it remains to establish (3.127). Note first that $\mathbb{P}_{ann}(\text{C1 happens}) = o(1)$ by (3.14) with $k=1$, $m_0 = 1$, $m_E = 0$ and $m = o(\sqrt{n})$ by Remark 3.5.2. Therefore, it is enough to prove that

$$\mathbb{P}^{\mathbb{T}_d}(|B_{\mathcal{C}_\circ^{h-\log^{-1} n}}(\circ, \lfloor (1/2-\varepsilon/2)\log_{\lambda_h} n\rfloor + a_n)| < n^{1/2-\varepsilon/8}) \to 1,$$

since if this event happens, less than $n^{1/2-\varepsilon/10}$ vertices and half-edges have been seen at step $\lfloor(1/2-\varepsilon/2)\log_{\lambda_h} n\rfloor$. By (3.33), it suffices to show that

$$\mathbb{P}^{\mathbb{T}_d}(|B_{\mathcal{C}_\circ^{h-\log^{-1} n}}(\circ, \lfloor(1/2-\varepsilon/2)\log_{\lambda_h} n\rfloor)| < n^{1/2-\varepsilon/7}) \to 1, \tag{3.128}$$

To do so, we first prove that for n large enough and every $\log\log n \leq k \leq (1/2-\varepsilon/2)\log_{\lambda_h} n$,

$$\mathbb{P}^{\mathbb{T}_d}\left(|\mathcal{Z}_k^{h-\log^{-1} n}| \geq n^{1/2-\varepsilon/6}\right) \leq e^{-Ck} + n^{-3}. \tag{3.129}$$

Let $\delta > 0$ such that $\lambda_h \leq \lambda_{h-\delta} \leq \lambda_h + \varepsilon/10$ (which is possible since $h' \mapsto \lambda_{h'}$ is decreasing and continuous, Proposition 3.3.4). By Proposition 3.3.8, there exists $C > 0$ (depending on ε) such that for n large enough, for every $\log\log n \leq k \leq (1/2-\varepsilon/2)\log_{\lambda_h} n$ and $a \geq h$,

$$\begin{aligned}
\mathbb{P}_a^{\mathbb{T}_d}\left(|\mathcal{Z}_k^{h-\log^{-1} n}| \geq n^{1/2-\varepsilon/5}\chi_{h-\delta}(a)\right) &\leq \mathbb{P}_a^{\mathbb{T}_d}\left(|\mathcal{Z}_k^{h-\delta}| \geq n^{1/2-\varepsilon/5}\chi_{h-\delta}(a)\right) \\
&\leq \mathbb{P}_a^{\mathbb{T}_d}\left(\log|\mathcal{Z}_k^{h-\delta}| \geq (1/2-\varepsilon/5)\lambda_h \log_{\lambda_h} n + \log\chi_{h-\delta}(a)\right) \\
&\leq \mathbb{P}_a^{\mathbb{T}_d}\left(\log|\mathcal{Z}_k^{h-\delta}| \geq (\lambda_{h-\delta} - \varepsilon/10)\frac{1/2-\varepsilon/5}{1/2-\varepsilon/2}k + \log\chi_{h-\delta}(a)\right) \\
&\leq \mathbb{P}_a^{\mathbb{T}_d}\left(\log|\mathcal{Z}_k^{h-\delta}| \geq (\lambda_{h-\delta} - \varepsilon/10)(1+3\varepsilon/5)k + \log\chi_{h-\delta}(a)\right) \\
&\leq \mathbb{P}_a^{\mathbb{T}_d}\left(\log|\mathcal{Z}_k^{h-\delta}| \geq \lambda_{h-\delta}(1-\varepsilon/10)(1+3\varepsilon/5)k + \log\chi_{h-\delta}(a)\right) \\
&\leq \mathbb{P}_a^{\mathbb{T}_d}\left(\log|\mathcal{Z}_k^{h-\delta}| \geq \lambda_{h-\delta}(1+\varepsilon/5)k + \log\chi_{h-\delta}(a)\right) \\
&\leq e^{-Ck}.
\end{aligned}$$

By Proposition 2.1 of [4], there exists $c > 0$ such that for all $h' \leq h_\star$ and $a \geq d-1$, one has $\chi_{h'}(a) \leq ca^{1-\log_{d-1}\lambda_{h'}} \leq ca$. Since $\chi_{h'}$ is continuous on $[h, +\infty)$ (Lemma 3.3.5), we have for n large enough $\max_{h \leq a \leq \log^2 n}\chi_{h-\delta}(a) < n^{\varepsilon/30}$, so that

$$\mathbb{P}^{\mathbb{T}_d}\left(|\mathcal{Z}_k^{h-\log^{-1} n}| \geq n^{1/2-\varepsilon/6}\right) \leq e^{-Ck} + \mathbb{P}^{\mathbb{T}_d}(\varphi_{\mathbb{T}_d}(\circ) \geq \log^2 n)$$

Using the exponential Markov inequality as in Lemma 3.2.5, we get $\mathbb{P}^{\mathbb{T}_d}(\varphi_{\mathbb{T}_d}(\circ) \geq \log^2 n) \leq n^{-3}$. This yields (3.129). Then, for n large enough, this implies

$$\begin{aligned}
\mathbb{P}^{\mathbb{T}_d}(|B_{\mathcal{C}_\circ^h}(\circ, \lfloor(1/2-\varepsilon/2)\log_{\lambda_h} n\rfloor)| \geq n^{1/2-\varepsilon/7}) \\
&\leq \mathbb{P}^{\mathbb{T}_d}(\exists k \in [\log\log n, (1/2-\varepsilon/2)\log_{\lambda_h} n], |\mathcal{Z}_k^{h-\log^{-1} n}| \geq n^{1/2-\varepsilon/6}) \\
&\leq \sum_{k=\lfloor\log\log n\rfloor}^{\lfloor(1/2-\varepsilon/2)\log_{\lambda_h} n\rfloor}(e^{-Ck} + n^{-3}) \\
&\leq 1/\log\log n.
\end{aligned}$$

(3.128) and the conclusion follow. □

3.8.4 The diameter: proof of (3.6)

Recall that $D_1^{(n)}$ is the diameter of $\mathcal{C}_1^{(n)}$. In Section 3.7, we have in fact proven that there exists a constant $K_{13} > 0$ such that for $\mathcal{E}_n := \{\forall x \in \mathcal{C}_1^{(n)},\ |B_{\mathcal{C}_1^{(n)}}(x, \lfloor K_{13} \log n \rfloor)| \geq K_{12} n\}$, we have

$$\mathbb{P}_{ann}(\mathcal{E}_n) \underset{n \to +\infty}{\longrightarrow} 1.$$

Namely, one can take $K_{13} = K_0 + 3\log^{-1} \lambda_h$.

Hence it is enough to show that on $\mathcal{E}_n$, $D_1^{(n)} \leq 6K_{12}^{-1} K_{13} \log n$, which will imply (3.6). We do this by a short deterministic argument.

Let $x_1 \in \mathcal{C}_1^{(n)}$. If $\partial B_{\mathcal{C}_1^{(n)}}(x_1, 2\lfloor K_{13} \log n \rfloor + 1) = \emptyset$, then

$$D_1^{(n)} \leq 4K_{13} \log n + 2.$$

Else, let $x_2 \in \partial B_{\mathcal{C}_1^{(n)}}(x_1, 2\lfloor K_{13} \log n \rfloor + 1)$. For $i = 1, 2$, we have

$$|B_{\mathcal{C}_1^{(n)}}(x_i, \lfloor K_{13} \log n \rfloor)| \geq K_{12} n \text{ and } B_{\mathcal{C}_1^{(n)}}(x_i, \lfloor K_{13} \log n \rfloor) \subseteq B_{\mathcal{C}_1^{(n)}}(x_1, 4\lfloor K_{13} \log n \rfloor).$$

Moreover, $B_{\mathcal{C}_1^{(n)}}(x_1, \lfloor K_{13} \log n \rfloor) \cap B_{\mathcal{C}_1^{(n)}}(x_2, \lfloor K_{13} \log n \rfloor) = \emptyset$. Thus, we have

$$|B_{\mathcal{C}_1^{(n)}}(x_1, 4\lfloor K_{13} \log n \rfloor)| \geq 2K_{12} n.$$

For $i \geq 2$, if $\partial B_{\mathcal{C}_1^{(n)}}(x_1, (3i-4)\lfloor K_{13} \log n \rfloor + 1) = \emptyset$, then $D_1^{(n)} \leq 2(3i-4)K_{13} \log n + 2$. Else, let $x_{i+1} \in \partial B_{\mathcal{C}_1^{(n)}}(x_1, (3i-4)\lfloor K_{13} \log n \rfloor + 1)$. As for $i = 2$, we get that

$$|B_{\mathcal{C}_1^{(n)}}(x_1, (3i-2)\lfloor K_{13} \log n \rfloor)| - |B_{\mathcal{C}_1^{(n)}}(x_1, (3i-5)\lfloor K_{13} \log n \rfloor)| \geq |B_{\mathcal{C}_1^{(n)}}(x_i, K_{13} \log n)| \geq K_{12} n.$$

But there are only n vertices in V_n. Therefore, $\partial B_{\mathcal{C}_1^{(n)}}(x_1, (3i_0 - 4)\lfloor K_{13} \log n \rfloor + 1) = \emptyset$ for some $i_0 \leq K_{12}^{-1}$, and

$$D_1^{(n)} \leq 6K_{12}^{-1} K_{13} \log n.$$

This shows (3.6).

Chapter 4

Cutoff for random walks on random lifts

Section 4.1 stems from the preprint [55], that has been accepted for publication by Annals of Probability. Sections 4.2 and 4.3 are original material of this thesis.

4.1 Generic cutoff for random lifts of weighted graphs

Abstract. We prove a cutoff for the random walk on random n-lifts of finite weighted graphs, even when the random walk on the base graph $\mathcal{G}$ of the lift is not reversible. The mixing time is w.h.p. $t_{mix} = h^{-1} \log n$, where h is a constant associated to $\mathcal{G}$, namely the entropy of its universal cover. Moreover, this mixing time is the smallest possible among all n-lifts of $\mathcal{G}$. In the particular case where the base graph is a vertex with $d/2$ loops, d even, we obtain a cutoff for a d-regular random graph, as did Lubetzky and Sly in [101] (with a slightly different distribution on d-regular graphs, but the mixing time is the same).

4.1.1 Introduction

The cutoff phenomenon

The way random walks converge to equilibrium on a graph is closely related to essential geometrical properties of the latter (such as the typical distance between vertices, its diameter, its expansion, the presence of traps or bottlenecks, etc.), giving an important motivation for studying mixing times.

For a Markov chain on a discrete state space Ω, with transition matrix P, that has an invariant distribution π, the **ε-mixing time from x** is

$$t_x(\varepsilon) := \inf\{t \geq 0, \, \|P^t(x, \cdot) - \pi\|_{TV} \leq \varepsilon\}, \tag{4.1}$$

where $\|\nu_1 - \nu_2\|_{TV} := \sup_{S\subseteq\Omega}(\nu_1(S) - \nu_2(S))$ is the total variation distance between the probability measures ν_1 and ν_2 on Ω. The **worst-case mixing time** $t_{max}(\varepsilon) := \sup\{t_x(\varepsilon),\, x \in \Omega\}$ is often the quantity of main interest. Other distances than the total variation distance might be considered. A straightforward computation shows that $t \mapsto \|P^t(x,\cdot) - \pi\|_{TV}$ is a non-increasing function, so that the definition of mixing time is relevant.

For a sequence of Markov chains $(\Omega_n, P_n, \pi_n)_{n\geq 0}$, there is **cutoff** when for all $\varepsilon, \varepsilon' \in (0,1)$, $t^{(n)}_{max}(\varepsilon)/t^{(n)}_{max}(\varepsilon') \to 1$ as $n \to +\infty$.

While the cutoff phenomenon remains far from being completely understood, first examples of it were given in the 1980s for different random walks on finite groups (see [63] or [14] on the symmetric group) or on spaces that can be factored into a n-product of a base space (such as $\mathbb{Z}_2^n$ in [9]), and this direction is still investigated nowadays (see for instance [83],[85] on random Cayley graphs of abelian groups).

A class of graphs where random walks mix fast and where cutoff is expected are the expander graphs. These are sequences $(G_n)_{n\geq 1}$ of graphs whose size goes to infinity (say G_n has n vertices) and whose isoperimetric constant is bounded away from 0: there exists $c > 0$ independent of n such that for any subset S of at most $n/2$ vertices of G_n, $|\partial S| \geq c|S|$, where $\partial S \subseteq S^c$ is the set of vertices adjacent to vertices of S. The very accessible survey of Hoory, Linial and Widgerson [87] provides a good overview of the study of these graphs. This expansion property entails the existence of a spectral gap (this implication is called the "Cheeger bound", see [17] for instance): the second largest eigenvalue of the transition matrix P_n of the SRW on G_n is bounded away from the largest one as $n \to +\infty$. It is classical that this spectral gap implies in turn that the SRW on G_n mixes in $O(\log n)$ steps.

The simplest expander model is the random d-regular graph (i.e. $G_n(d)$ is chosen uniformly among graphs with n vertices having all degree d). Friedman [45] proved in 2002 that w.h.p., $G_n(d)$ almost achieves the largest possible spectral gap, while Lubetzky and Sly [101] proved in 2008 that the SRW and the NBRW (Non-Backtracking Random Walk, i.e. a SRW conditioned at each step on not going back along the edge it has just crossed) on $G_n(d)$ admit a cutoff. Several papers followed on cutoffs for other sparse graphs: see for instance [29] for the SRW on the largest component of a supercritical Erdős-Rényi random graph, [27] for the NBRW on a configuration model, or [84] for a generic perspective.

Very recently, there has been increasing interest in mixing times on dynamical graphs (typically, edges are re-sampled at random at a given rate), when the mixing time profile is already well-known on a static version of the graph (for instance [24], [128] and [51]).

A natural way of combining the "product of a base space" and the "expanding sparse graph" perspectives for cutoff is to consider random walks on random n-lifts of a fixed graph $\mathcal{G}$, that we define now.

Random walks on weighted graphs

For a multigraph $\mathcal{G}$ (multiple edges and multiple loops are allowed), denote $V_\mathcal{G}$ its **vertex set** and $E_\mathcal{G}$ its **edge set**. Every edge $e \in E_\mathcal{G}$ gives rise to two opposite oriented edges. Denote $\overrightarrow{E}_\mathcal{G}$ the set of oriented edges of $\mathcal{G}$. For each $\overrightarrow{e} \in \overrightarrow{E}_\mathcal{G}$, note $\overrightarrow{e}^{-1}$ its opposite. We study weighted random walks by giving to each $\overrightarrow{e} \in \overrightarrow{E}_\mathcal{G}$ a nonnegative weight $w(\overrightarrow{e})$, so that

- every $e \in E_\mathcal{G}$ has at least one orientation with positive weight,
- for all $u \in V_\mathcal{G}$, the sum of the weights of the oriented edges going out of u is 1.

We define **the random walk (RW) on** $\mathcal{G}$ as a discrete-time Markov Chain $(X_t)_{t\geq 0}$ on $V_\mathcal{G}$ with transition matrix $P_\mathcal{G}$ such that for all $u, v \in V_\mathcal{G}$,

$$P_\mathcal{G}(u,v) = \frac{1}{2}\mathbf{1}_{\{u=v\}} + \frac{1}{2}\sum_{\overrightarrow{e}:u\to v} w(\overrightarrow{e}),$$

where "$\overrightarrow{e} : u \to v$" means that the **initial vertex** of $\overrightarrow{e}$ is u and its **end vertex** is v.

Random lifts

Fix now a finite multigraph $\mathcal{G}$. A n**-lift of** $\mathcal{G}$ is a graph $\mathcal{G}_n$ with vertex set $V_{\mathcal{G}_n} := V_\mathcal{G} \times [n]$, and edge set $E_{\mathcal{G}_n}$ as follows: fix for each $e \in E_\mathcal{G}$ an arbitrary ordering (u, v) of its endpoints and a permutation $\sigma_e \in \mathfrak{S}_n$, and draw the edges $\{u_i, v_{\sigma_e(i)}\}$ for all $1 \leq i \leq n$ (see Figure 1). Say that u (resp. v) is the **type** of u_i (resp. $v_{\sigma(i)}$) and that e is the **type** of $\{u_i, v_{\sigma(i)}\}$.
When the σ_e's are uniform independent permutations, $\mathcal{G}_n$ is a **random** n**-lift of** $\mathcal{G}$.
For simplicity of the notations, write V_n, E_n and $\overrightarrow{E}_n$ for the vertex set, edge set and oriented edge set of $\mathcal{G}_n$. Define as previously the **type** of an element of $\overrightarrow{E}_n$ as the corresponding oriented edge of $\overrightarrow{E}_\mathcal{G}$, and give to each $\overrightarrow{e} \in \overrightarrow{E}_n$ the weight of its type.
Denote π the invariant probability measure of the RW on $\mathcal{G}$ (if it exists). Note that the RW on $\mathcal{G}_n$ has an invariant measure π_n such that $\pi_n((x,i)) = \pi(x)/n$ for all $x \in V_\mathcal{G}$, $i \in [n]$. Write P_n for the transition matrix of the RW on $\mathcal{G}_n$. Let π_{min} and π_{max} be the smallest and largest values taken by π on $V_\mathcal{G}$. Finally, we denote $w_{min} > 0$ the smallest positive weight in $\mathcal{G}$ and Δ the largest degree in $\mathcal{G}$ (the **degree** of a vertex being the number of edges attached to it).
The graph structure of random lifts has been studied since the early 2000s (see [20], [21], [22] and [99]). In particular, it is proved in [20] that random n-lifts are expanders w.h.p. as n goes to infinity, as long as $\mathcal{G}$ has at least two cycles. More recently, spectral properties of lifts have been investigated (see for instance [8], [66], [102]): Bordenave [45] generalized Friedman's theorem to the NBRW on random n-lifts of a finite graph, then Bordenave and Collins [46] established a similar result for the SRW. Bordenave and Lacoin [47] proved that if the RW associated to $\mathcal{G}$ is reversible, and if the invariant measure is uniform, then the RW on $\mathcal{G}_n$ admits a cutoff, with a mixing time in $h^{-1}\log n + o(\log n)$ steps, for some constant h (the "entropy") depending on $\mathcal{G}$.

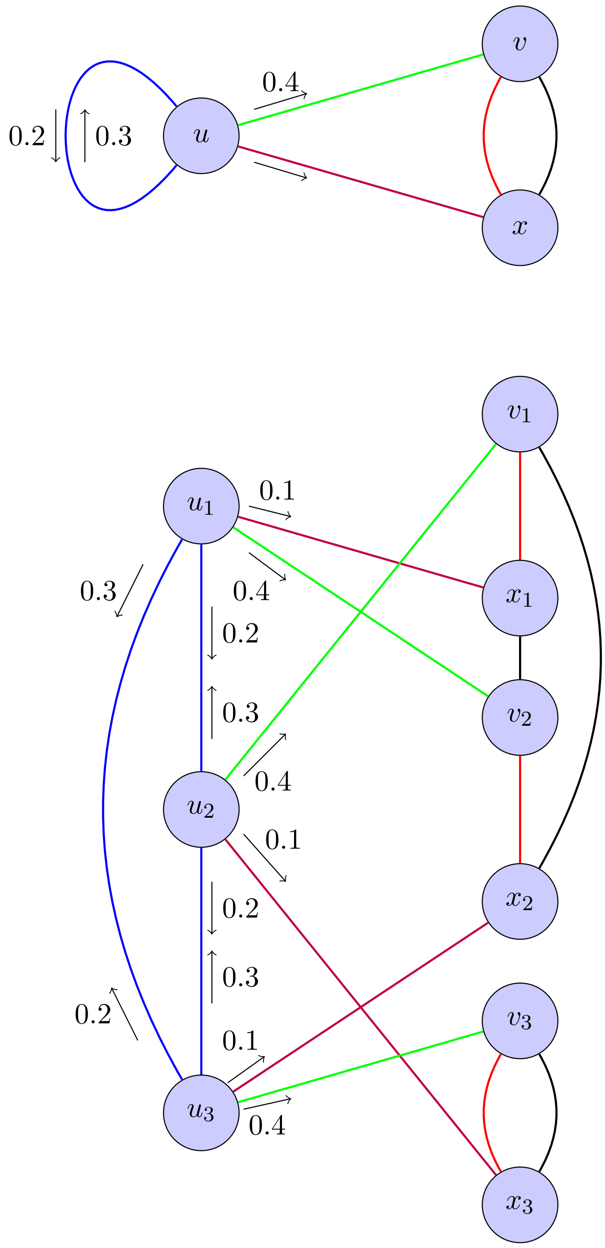

Figure 1: a weighted graph $\mathcal{G}$ and a 3-lift of $\mathcal{G}$ (not all weights are written on the picture). $\sigma_{\{u,u\}} = (2\,3\,1)$, $\sigma_{\{u,v\}} = (2\,1)\,(3)$, $\sigma_{\{u,x\}} = (1\,3\,2)$, $\sigma_{\{v,x,\text{red}\}} = (1)\,(2)\,(3)$, $\sigma_{\{v,x,\text{black}\}} = (2\,1)\,(3)$.

Results

We characterize all (finite) irreducible graphs $\mathcal{G}$ such that there is w.h.p. cutoff for the random walk on a random n-lift of $\mathcal{G}$, and we prove that the cutoff window is of order $\sqrt{\log n}$.
We introduce the following assumptions:

A.1 the RW on $\mathcal{G}$ is irreducible,

A.2 **[Two-cycles property]** $\mathcal{G}$ has at least two oriented cycles which are not each other's inverse, where an **oriented cycle of length** $m \geq 1$ is a cyclic order $C = (\overrightarrow{e}_1, \dots, \overrightarrow{e}_m)$ of m oriented edges with positive weight such that the end vertex of $\overrightarrow{e}_i$ is the initial vertex of $\overrightarrow{e}_{i+1}$ and $\overrightarrow{e}_i \neq \overrightarrow{e}_{i+1}^{-1}$ for all $i \pmod m$, and the **inverse** of this cycle is the cycle $C^{-1} = (\overrightarrow{e}_m^{-1}, \dots, \overrightarrow{e}_1^{-1})$.

Theorem 4.1.1. *Suppose that $\mathcal{G}$ satisfies **A.1** and **A.2**, and that $\mathcal{G}_n$ is a uniform random lift of $\mathcal{G}$. For any $\varepsilon \in (0,1)$, there exists $K(\varepsilon) > 0$ such that if $t_{max}^{(n)}(\varepsilon)$ is the worst-case mixing time of the RW on $\mathcal{G}_n$, then w.h.p. on the realization of $\mathcal{G}_n$ as $n \to +\infty$,*

$$|t_{max}^{(n)}(\varepsilon) - h^{-1} \log n| \leq K(\varepsilon)\sqrt{\log n} \tag{4.2}$$

the constant $h > 0$ being the entropy of the universal cover of $\mathcal{G}$ (see Section 4.1.3).

Let

$$\Phi(\lambda) := \frac{1}{\sqrt{2\pi}} \int_{\lambda}^{+\infty} e^{-\frac{u^2}{2}} du \tag{4.3}$$

for $\lambda \in \mathbb{R}$ be the tail distribution of the standard normal. The following lower bound shows that the mixing time of Theorem 4.1.1 is almost the smallest possible among all n-lifts of $\mathcal{G}$:

Proposition 4.1.2. *There exists $\sigma > 0$, uniquely depending on $\mathcal{G}$, such that for any $\varepsilon \in (0,1)$, for any arbitrary sequence $(\mathcal{G}_n)_{n\geq 1}$ of n-lifts of $\mathcal{G}$,*

$$\liminf_{n\to+\infty} \frac{t_{min}^{(n)}(\varepsilon) - h^{-1}\log n}{\sqrt{\log n}} \geq \sigma\Phi^{-1}(\varepsilon), \tag{4.4}$$

where $t_{min}^{(n)}(\varepsilon) := \min_{x\in V_n} t_x^{(n)}(\varepsilon)$ is the best-case ε-mixing time (ie, the shortest mixing time among all possible starting vertices).

Assumption **A.2** is necessary in Theorem 4.1.1: remark that if $\mathcal{G}$ has at most one oriented cycle, $\inf_{x\in V_n} \pi_n (\{y \in V_n,$ there is no oriented path from x to $y\})$ is w.h.p. bounded away from 0.
We conjecture that the lower bound of Proposition 4.1.2 is optimal, so that the cutoff window would have a Gaussian profile. This was established for the SRW on the random d-regular graph $G_n(d)$ in [101]. In this case, there is randomness for the speed of the walk (since it can backtrack), but the degree of the vertices met by the walk is constant. Conversely, for the NBRW on the configuration model whose cutoff window also exhibits this behaviour [27], there is randomness for the degrees met by the walk, but not for the speed. In our setting, as for the SRW on the configuration model [29], both the environment and the speed of the walk might vary, and the result of this combination is not clear.
Theorem 4.1.1 is also true for a **lazy random walk** on V_n with any **holding probability** $\alpha \in (0,1)$, i.e. for all $u, v \in V_n$, the transition matrix is $P_n^{(\alpha)}(u,v) = \alpha \mathbf{1}_{\{u=v\}} + (1-\alpha)\sum_{\overrightarrow{e}:u\to v} w(\overrightarrow{e})$ (hence, the RW we defined in Section 4.1.1 and that we will study throughout this paper is lazy with holding probability 1/2). This gives a new value of the entropy:

$$h_\alpha = \frac{h}{2(1-\alpha)}. \tag{4.5}$$

The question whether this holds in the case $\alpha = 0$ remains unsolved, see Appendix 2 (Section 4.1.7) for a discussion.
One might also investigate to what extent Theorem 4.1.1 holds when $\mathcal{G}$ changes with n. It is proven in [101] that there is still cutoff for the RW on $G_n(d)$ if $d = n^{o(1)}$.

Examples

Our setting is very general, since it includes lifts of any finite Markov chain with positive holding probability. We highlight two special cases below.

Random walks on the d-regular random graph
We can recover an approximate version of the result of [101] for the RW on d-regular graphs, when d is even: in the very particular case when $\mathcal{G}$ consists of a single vertex and $d/2$ loops having weight $1/d$ on both orientations, a random n-lift of $\mathcal{G}$ is a random d-regular multigraph (but its distribution is neither that of a uniform d-regular multigraph, nor that of $G_n(d)$). Our results allow us to conjecture the cutoff for the SRW (which is a RW with holding probability $\alpha = 0$): Proposition 4.1.2 holds for $\alpha = 0$ and gives the lower bound,. The upper bound would come from Theorem 4.1.1 with $\alpha > 0$ arbitrarily small, but it is not clear that the arguments would work for $\alpha = 0$ (see Appendix 4.1.7). One gets $h_0 = \frac{(d-2)\log(d-1)}{d}$ (tools for its computation are in Section 4.1.3). This is exactly the value of h in Theorem 1 of [101] for the SRW on $G_n(d)$. Their theorem states in addition that the cutoff window is of order $\sqrt{\log n}$ with a Gaussian profile, hence corresponding to our lower bound.

Cutoff for non-Ramanujan graphs
It was recently proven that on every sequence of d-regular weakly Ramanujan graphs, the SRW admits a cutoff [100] (a sequence G_n of d-regular graphs is said to be **weakly Ramanujan** whenever for all $\varepsilon > 0$, every eigenvalue of the adjacency matrix of G_n is either $\pm d$ or in $[-2\sqrt{d-1} - \varepsilon, 2\sqrt{d-1} + \varepsilon]$ for n large enough). This was even extended to graphs having $n^{o(1)}$ eigenvalues anywhere in $(-d + \varepsilon', d - \varepsilon')$ for an arbitrary $\varepsilon' > 0$.
Theorem 4.1.1 gives an alternative proof for the existence of sequences of non weakly Ramanujan graphs having a cutoff. Indeed, take for $\mathcal{G}$ a connected d-regular graph which is not Ramanujan. One computes easily that all eigenvalues of $\mathcal{G}$ are also eigenvalues of $\mathcal{G}_n$, so that $\mathcal{G}_n$ is not weakly Ramanujan.

Tools and reasoning

The graph we study has locally few cycles, so that the behaviour of the RW on $\mathcal{G}_n$ is closely linked to that of a RW on its **universal cover** $(\mathcal{T}_{\mathcal{G}}, \circ)$, the infinite rooted tree obtained from $\mathcal{G}$ by "unfolding" all its non-backtracking paths starting at a given distinguished vertex $\circ$, the root. A **non-backtracking path** is an oriented path $(\overrightarrow{e}_1, \dots, \overrightarrow{e}_m)$ such that $\overrightarrow{e}_{i+1} \neq \overrightarrow{e}_i^{-1}$

for all $i \leq m-1$. Two vertices in $\mathcal{T}_\mathcal{G}$ are neighbours if and only if the non-backtracking path of one of them is the non-backtracking path of the other without its last edge, see Figure 2 for an example.
We will write abusively $\mathcal{T}_\mathcal{G}$ instead of $(\mathcal{T}_\mathcal{G}, \circ)$ when the root is irrelevant.

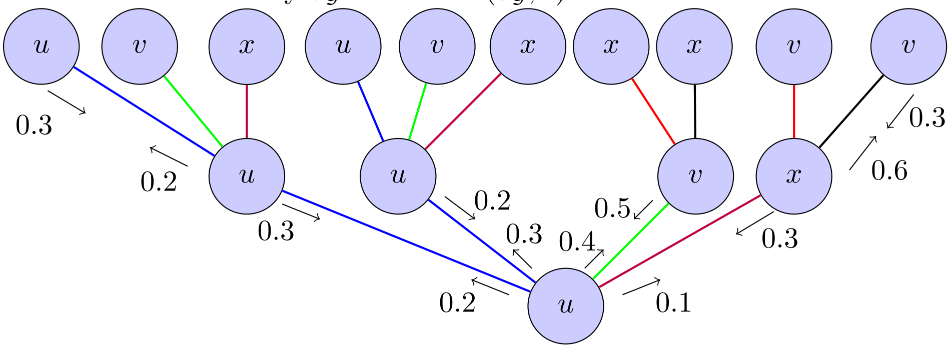

Figure 2: the first levels of the universal cover of $\mathcal{G}$ in Figure 1 (not all weights are on the picture), rooted at u. Note that two non-backtracking paths start from u, since the blue edge in $\mathcal{G}$ gives rise to two oriented edges opposite to each other.

This object, also called "periodic tree", has been thoroughly studied since the 1990s. We postpone a precise history to Section 4.1.3. Our main references are an article on trees with finitely many cone types [117], and another on regular languages [81].
Let $(\mathcal{X}_t)_{t\geq 0}$ be a RW on $(\mathcal{T}_\mathcal{G}, \circ)$ starting at the root. A key observation is that this RW is transient. This is intuitive, since $(\mathcal{T}_\mathcal{G}, \circ)$ has an exponential growth by **A.2** (that is, for some $C > 1$ and all R large enough, there are more than C^R vertices at distance R from the root) and has a "regular" structure for the RW (**A.1** guarantees the existence of an invariant measure). Thus, we can define its loop-erased trace, or "ray to infinity" $\xi := (\xi_t)_{t\geq 0}$, ξ_t being the last vertex visited by the RW at distance t of the root.
Let $W_t := -\log W(\mathcal{X}_t)$ where for $x \in (\mathcal{T}_\mathcal{G}, \circ)$, $W(x) = \mathbb{P}(x \in \xi)$. The following central limit theorem sums up almost all the information we need on $\mathcal{T}_\mathcal{G}$.

Theorem 4.1.3 (CLT for the weight on $\mathcal{T}_\mathcal{G}$). *There exist two constants $h_{\mathcal{T}_\mathcal{G}} > 0, \sigma_{\mathcal{T}_\mathcal{G}} \geq 0$, only depending on $\mathcal{T}_\mathcal{G}$, such that*

$$\frac{W_t - h_{\mathcal{T}_\mathcal{G}} t}{\sqrt{t}} \overset{law}{\to} \mathcal{N}(0, \sigma^2_{\mathcal{T}_\mathcal{G}}), \tag{4.6}$$

with the convention that $\mathcal{N}(0,0) = \delta_0$ is the Dirac distribution in 0.

The proof relies on the regularity of the structure of $\mathcal{T}_\mathcal{G}$, that allows us to cut the trajectory of $(\mathcal{X}_t)_{t\geq 0}$ into i.i.d. intervals between regeneration times at oriented edges of a certain fixed type. This gives us almost directly a proof of Proposition 4.1.2. The proof of Theorem 4.1.1 proceeds in three steps:

a) we couple $(\mathcal{X}_t)_{t\geq 0}$ to a RW $(X_t)_{t\geq 0}$ on $\mathcal{G}_n$, imitating the analogous construction in [29] for the configuration model. This coupling is viable as long as (X_t) does not meet cycles.

We can ensure this almost until the mixing time, for most starting points of the RW in $\mathcal{G}_n$. We stress that the RW on $(\mathcal{T}_{\mathcal{G}}, \circ)$ is strongly "localized" around ξ:

Proposition 4.1.4 (Ray localization).

$$\forall R, t \geq 1, \, \mathbb{P}(\xi \cap B(\mathcal{X}_t, R) = \emptyset) \leq C_1 \exp(-C_2 R),$$

where for all $y \in (\mathcal{T}_{\mathcal{G}}, \circ)$ and $r \geq 0$, $B(y, r)$ is the set of vertices y' such that there is an oriented path of length $\leq r$ from y to y'. This crucial observation allows us to reveal a limited number of edges while coupling RWs on $(\mathcal{T}_{\mathcal{G}}, \circ)$ and $\mathcal{G}_n$, hence reducing the probability to meet a cycle. This leads to an "almost mixing" of the RW on $\mathcal{G}_n$ after $h^{-1} \log n + O(\sqrt{\log n})$ steps: the mass of $P_n^t(X_0, \cdot)$ is concentrated on values of order $n^{-1} e^{O(\sqrt{\log n})}$ for some $t = h^{-1} \log n + O(\sqrt{\log n})$.

Corollary 4.1.5 (Almost mixing). *Let $\varepsilon, K > 0$ and $a < 0$. If $-a$ and K are both large enough (depending on $\mathcal{G}$ and ε), then for n large enough, with probability at least $1 - \varepsilon$, $\mathcal{G}_n$ is such that for all $x \in V_n$,*

$$\nu_n(V_n) \geq 1 - \varepsilon,$$

where $\nu_n(x') := P_n^{t'_n}(x, x') \wedge \frac{\exp(K\sqrt{\log n})}{n}$ for all $x' \in V_n$, and $t'_n := h^{-1} \log n + a\sqrt{\log n}$.

b) As in [29], a spectral argument relying on the good expanding properties of random lifts (generalizing a little the result of [20]) allows us to make the last jump until the mixing time. We emphasize the fact that this spectral property holds even if the RW on $\mathcal{G}$ is not reversible.

c) Finally, we extend the mixing to every starting point for the RW in $\mathcal{G}_n$, proving that $(X_t)_{t \geq 0}$ quickly reaches a vertex to which we can apply a). We adapt the technique in [26], which was originally designed for the configuration model considered in [29].

Plan

We start with basic but essential properties in Section 4.1.2. We study the universal cover of $\mathcal{G}$ in Section 4.1.3, and prove Proposition 4.1.2 and Theorem 4.1.1 in Section 4.1.4, under some additional assumptions on $\mathcal{G}$ introduced in Section 4.1.2. We show in Section 4.1.5 that those assumptions on $\mathcal{G}$ are not necessary. We discuss the computation of the constant h in Section 4.1.6, and the case $\alpha = 0$ in Section 4.1.7.

4.1.2 Basic properties

Three Lemmas

The next property is an essential tool for building $\mathcal{G}_n$ while exploring it via a walk on its vertices. It is analogous to the classical construction of a configuration model.

Lemma 4.1.6 (Edge by edge construction). *A random lift $\mathcal{G}_n$ of $\mathcal{G}$ can be generated sequentially as follows. Consider n copies of $\mathcal{G}$, split every edge e in two half-edges, respectively attached to the first and second vertex of e. Define their respective* ***type*** *as the type of the orientation of e starting at the first (resp. second) vertex. Perform the following operations:*

1. *pick arbitrarily an unmatched half-edge, say of type $\overrightarrow{e} \in \overrightarrow{E}_{\mathcal{G}}$,*
2. *match it with another unmatched half-edge, uniformly chosen among those of type $\overrightarrow{e}^{-1}$,*
3. *repeat steps 1. and 2. until all half-edges are matched.*

Proof. One checks easily that the obtained structure is a n-lift of $\mathcal{G}$, and that all permutations along edges of $\mathcal{G}$ are uniform and together independent, as in the definition. □

Random walks on lifts admit a natural projection property (whose proof is straightforward):

Lemma 4.1.7 (Projection of the lift). *Fix $n \in \mathbb{N}$. Let $\mathcal{G}_n$ be a n-lift of $\mathcal{G}$ and let $(X_t)_{t\geq 0}$ be a RW on $\mathcal{G}_n$. Let $(\overline{X}_t)_{t\geq 0}$ be the projection of $(X_t)_{t\geq 0}$ on $\mathcal{G}$, obtained by mapping X_t to its type for all $t \geq 0$.*
Then $(\overline{X}_t)_{t\geq 0}$ is a RW on $\mathcal{G}$.

Hence, walks on lifts inherit much of the structure of walks on the base graph.
It is well known that for a Markov chain $(X_t)_{t\geq 0}$ on a finite set Ω with invariant measure π, for any $u \in \Omega$, $\lim_{t\to+\infty} \mathbb{P}(X_t = u) = \pi(u)$ provided that the chain is aperiodic. We finally state a CLT refining this ergodic property (and not requiring aperiodicity):

Lemma 4.1.8 (CLT for Markov chains). *Let f be a function from Ω to $\mathbb{R}$. For $n \in \mathbb{N}$, let $S_n := \sum_{t=0}^{n-1} f(X_t)$. Let $m := \sum_{u\in\Omega} f(u)\pi(u)$. Fix $u \in \Omega$, $X_0 = u$ a.s. and let τ_u be the first hitting time of u after 0. Let*

$$v := \pi(u)\mathbb{E}\left[\left(\sum_{t=0}^{\tau(u)-1} f(X_t) \ - \pi(u)\mathbb{E}[\sum_{t=0}^{\tau(u)-1} f(X_t)]\tau_u\right)^2\right].$$

Then v does not depend on the choice of u and

$$\frac{S_n - mn}{\sqrt{n}} \to \mathcal{N}(0, v),$$

where $\mathcal{N}(0,0)$ is the Dirac mass in 0. Moreover, if $v = 0$, $|S_n - mn|$ is bounded.

This is a direct application of Theorem 16.1 and Corollary 16.1 (p.94) in [54].

Additional assumptions on $\mathcal{G}$

Let us make the following additional assumptions on $\mathcal{G}$ (we will prove in that they are in fact not necessary, in Section 4.1.5):

A.3 All oriented edges have a positive weight,

A.4 every oriented edge lies on an oriented cycle.

For all $u \in \mathcal{G}$ and $R \geq 1$, let $\partial B(u,R) := B(u,R) \setminus B(u,R-1)$ be the **boundary** of $B(u,R)$. When **A.3** holds on $\mathcal{G}$, every non-oriented path gives rise to two oriented paths of positive weight (opposite to each other). We can define the **distance** $d(x,y)$ between two vertices x, y as the length of the shortest path with positive weight from x to y (and d is indeed a metric). We state those definitions for $\mathcal{G}$ but might use them for other graphs.

4.1.3 Study of the universal cover

The goal of this Section is to prove Theorem 4.1.3. Fix an arbitrary vertex $v_* \in V_{\mathcal{G}}$ and an arbitrary oriented edge $\overrightarrow{e}_*$ going out of v_* (this choice has no importance for the sequel), and root the universal cover of $\mathcal{G}$ at a vertex $\circ$ with label v_*.

Definitions for labelled rooted trees

Label each vertex $x \in V_{\mathcal{T}_{\mathcal{G}}}$ by the vertex $v \in V_{\mathcal{G}}$ such that the non-backtracking path in $\mathcal{G}$ starting at v_*, and corresponding to the path from $\circ$ to x in $(\mathcal{T}_{\mathcal{G}}, \circ)$, terminates at v (see Figure 2). Give similarly a **label** in $E_{\mathcal{G}}$ (resp. $\overrightarrow{E}_{\mathcal{G}}$) to each edge (resp. oriented edge) of $(\mathcal{T}_{\mathcal{G}}, \circ)$. In the literature, $(\mathcal{T}_{\mathcal{G}}, \circ)$ is sometimes called the **directed cover** of $\mathcal{G}$, or **periodic tree** arising from $\mathcal{G}$. Remark that the universal cover of an irreducible n-lift of $\mathcal{G}$ is also $(\mathcal{T}_{\mathcal{G}}, \circ)$.
Due to **A.3**, the RW on $(\mathcal{T}_{\mathcal{G}}, \circ)$ is an irreducible Markov chain with invariant measure $\widetilde{\pi}$ defined as follows: for all $x \in V_{\mathcal{T}_{\mathcal{G}}}$ with label u, $\tilde{\pi}(x) = \pi(u)$. Denote $P_{\mathcal{T}_{\mathcal{G}}}$ its transition matrix.
An **isomorphism** ϕ between two rooted trees $(T, \circ)$ and $(T', \circ')$ is a bijection between the vertices of $(T, \circ)$ and those of $(T', \circ')$ such that $\phi(\circ) = \circ'$ and such that there is an edge between x_1 and x_2 in $(T, \circ)$ if and only if there is an edge between $\phi(x_1)$ and $\phi(x_2)$ in $(T', \circ')$.

Lemma 4.1.9 (**Projection of the cover**). *If one projects $(\mathcal{T}_{\mathcal{G}}, \circ)$ on $\mathcal{G}$ by mapping each vertex, edge and oriented edge to its label, then the projection of a RW on $(\mathcal{T}_{\mathcal{G}}, \circ)$ is a RW on $\mathcal{G}$.*

Remark that this result is analogous to Lemma 4.1.7. Its proof is straightforward. Note that there exists at most $|V|$ distinct rooted trees $(\mathcal{T}_{\mathcal{G}}, \circ)$ up to isomorphism. Indeed, two vertices in $V_{\mathcal{T}_{\mathcal{G}}}$ with the same label induce isomorphic rooted trees.
For $x \in V_{\mathcal{T}_{\mathcal{G}}}$, let $he(x) := d(\circ, x)$ be the **height** of x in $(\mathcal{T}_{\mathcal{G}}, \circ)$. The **(rooted) subtree $\mathcal{T}_x$ from x in** $(\mathcal{T}_{\mathcal{G}}, \circ)$ is rooted at x, has vertex set $V_x := \{y \in V_{\mathcal{T}_{\mathcal{G}}},\ x \text{ is on any path from } \circ \text{ to } y\}$, and the same edges, weights and labels as $(\mathcal{T}_{\mathcal{G}}, \circ)$ on V_x. Note as previously that there exist finitely many such subtrees up to isomorphism. V_x is the **offspring of** x. If $y \in V_x$, it is a **descendant** of x **at generation** $he(y) - he(x)$, and x is the $(he(y) - he(x))$**-ancestor** of y. If in addition $he(y) = he(x) + 1$, y is a **child** of x and x is its **parent**. The **height-R level of** $\mathcal{T}_x$ denotes $\partial B(x,R)$, and $B(x,R) \cap V_x$ (resp. $\partial B(x,R) \cap V_x$) is also called the **offspring of** x **up to generation R (resp. offspring of generation R, or R-offspring)**.

An oriented edge in a rooted tree is **upward** if the height of its initial vertex is smaller than the height of its end vertex, **downward** else. Its **height** is the height of its initial vertex. An oriented path of upward (resp. downward) edges is an **upward** (resp. **downward**) **path**. In a tree, there is at most one edge between two vertices x, y. We will denote $\{x, y\}$, (x, y) and (y, x) the corresponding edge and oriented edges.

A little history and plan of Section 4.1.3

Lyons [105] and Takacs [133] have studied RWs on rooted periodic trees arising from simple graphs, with weights corresponding to the SRW with a positive or negative bias towards the root. Thus, this intersects our setting only when the bias vanishes, the RW being a SRW (hence, it is reversible). The more general case of trees with finitely many cone types has been studied by Nagnibeda and Woess [117] in 2002: these rooted weighted trees have finitely many subtrees up to isomorphism (periodic trees obviously have finitely many cone types, whereas the converse is not true). Their work relies on a fine understanding of the Green function initiated in [116], which satisfies a finite system of (non-linear) equations, due to the repetitive structure of the tree. Among others, they give a transience criterion for the RW in terms of the eigenvalues of a matrix associated to the tree, and obtain a CLT for the **rate of escape** (or **speed**, i.e. $\lim_{t\to+\infty} he(\mathcal{X}_t)/t$) when it exists) in the transient case. Similar formulas for Green functions were derived around the same time by Lalley [94] in the broader setting of regular languages. Gilch [81] later gave a formula for the entropy of the RW on regular languages, i.e. a law of large numbers for $(\log P^k(\mathcal{X}_0, \mathcal{X}_k))_{k\geq 0}$ where P is the transition matrix of the RW.

We extend slightly some of those results in the setting of periodic trees. We first give in Section 4.1.3 a simple transience criterion for the RW on the universal cover in terms of the base graph (Proposition 4.1.10), which we have not found in the literature. In particular, under assumptions **A.1**, **A.2**, **A.3** and **A.4** on $\mathcal{G}$, the RW on $(\mathcal{T}_{\mathcal{G}}, \circ)$ is transient (Corollary 4.1.12). A crucial argument from [78] is that whether the RW is reversible or not is irrelevant as soon as the RW is irreducible and there exists an invariant measure (in our case, $\widetilde{\pi}$). Hence, we can apply a classical transience criterion for reversible RWs (see [106]), noticing that $|B(\circ, R)|$ grows exponentially with R by **A.2**.

Thus, the loop-erased trace $(\xi_t)_{t\geq 0}$ of the RW is an infinite injective path. It is a Markov chain on $(\mathcal{T}_{\mathcal{G}}, \circ)$ with an easy description (Proposition 4.1.15). We can then define the **entropic weight** of a vertex $x \in V_{\mathcal{T}_{\mathcal{G}}}$ as the probability $W(x)$ that x is in ξ if the RW starts at $\circ$. The regularity of the structure of $(\mathcal{T}_{\mathcal{G}}, \circ)$ gives us a CLT for $(\log W(\xi_t))_{t\geq 0}$ (Corollary 4.1.16). Note that in the analogous Proposition 3 of [29] (in this case, the graph is locally a Galton-Watson tree), there is only a law of large numbers and a domination for the variance, and this prevents already to determine the profile of the cutoff window.

It remains to prove, in Section 4.1.3, that the RW on $(\mathcal{T}_{\mathcal{G}}, \circ)$ has a positive speed. Precise estimates on the Green function obtained by Lalley [94] and Nagnibeda and Woess [117] show in particular that for fixed $x, y \in \mathcal{T}_{\mathcal{G}}$, $P^n_{\mathcal{T}_{\mathcal{G}}}(x,y)$ decays exponentially with n. We prove that there exist random times (τ_i) with exponential moments (Proposition 4.1.19) such that in $(\mathcal{T}_{\mathcal{G}}, \circ)$, $he(\mathcal{X}_{\tau_i}) \geq i$ and $he(\mathcal{X}_t) > he(\mathcal{X}_{\tau_i})$ for all $t > \tau_i$. Moreover, the trajectory of the RW after τ_i is independent of the trajectory until τ_i. Hence we can decompose the RW into i.i.d. excursions whose durations have exponential moments. This regularity allows us to prove that $(he(\mathcal{X}_t))_{t\geq 0}$ and $(\log W(\mathcal{X}_t))_{t\geq 0}$ admit a CLT with nonzero mean (Theorem 4.1.3 and Proposition 4.1.20).

Transience of the RW on $\mathcal{T}_{\mathcal{G}}$

In this section, we give necessary and sufficient conditions on $\mathcal{G}$ for the RW on $(\mathcal{T}_{\mathcal{G}}, \circ)$ to be transient, and we prove Proposition 4.1.4.

Proposition 4.1.10. *Suppose that **A.1** holds for $\mathcal{G}$. Then the RW on $(\mathcal{T}_{\mathcal{G}}, \circ)$ is transient if and only if:*

- *either $\mathcal{G}$ satisfies Assumption **A.2**,*

- *or $\mathcal{G}$ has one oriented cycle $(\overrightarrow{e}_1, \ldots, \overrightarrow{e}_m)$ for some $m \geq 1$ such that $w(\overrightarrow{e}_1) \times \ldots \times w(\overrightarrow{e}_m) \neq w(\overrightarrow{e}_m^{-1}) \times \ldots \times w(\overrightarrow{e}_1^{-1})$.*

Proof. If $\mathcal{G}$ has no oriented cycle, then $(\mathcal{T}_{\mathcal{G}}, \circ)$ is finite and isomorphic to $\mathcal{G}$, and the RW is recurrent.
From now on, assume that $\mathcal{G}$ has at least one oriented cycle. Lemma 4.1.11 below deals with the case where **A.4** does not hold for $\mathcal{G}$. Hence, suppose now the contrary.
If **A.2** does not hold, then by **A.4**, $\mathcal{G}$ is reduced to a cycle. Denote $C = (\overrightarrow{e}_1, \ldots, \overrightarrow{e}_m)$ one orientation of this cycle. $(\mathcal{T}_{\mathcal{G}}, \circ)$ is a line, and the transition probabilities along this line are periodic and are given by the $w(\overrightarrow{e}_i)$'s and $w(\overrightarrow{e}_i^{-1})$'s. It is recurrent if and only if $w(\overrightarrow{e}_1) \times \ldots \times w(\overrightarrow{e}_m) = w(\overrightarrow{e}_m^{-1}) \times \ldots \times w(\overrightarrow{e}_1^{-1})$, see Woess [137] for a proof and a detailed study of this one-cycle case. In a nutshell, one can compute that the probability that a RW on $\mathcal{G}$ starting at the initial vertex x of $\overrightarrow{e}_1$ runs through C before running through C^{-1} is

$$\frac{w(\overrightarrow{e}_1) \times \ldots \times w(\overrightarrow{e}_m)}{w(e_1) \times \ldots \times w(\overrightarrow{e}_m) + w(\overrightarrow{e}_m^{-1}) \times \ldots \times w(\overrightarrow{e}_1^{-1})}.$$

If $(X_t)_{t\geq 0}$ is a RW on $(\mathcal{T}_{\mathcal{G}}, \circ)$, the law of its visits to the copies of x is that of a RW on $\mathbb{Z}$ (where each integer represents a copy of x, and each interval between two consecutive integers represents a copy of C) with transition probabilities

$$p(i, i+1) = 1 - p(i, i-1) = \frac{w(\overrightarrow{e}_1) \times \ldots \times w(\overrightarrow{e}_m)}{w(e_1) \times \ldots \times w(\overrightarrow{e}_m) + w(\overrightarrow{e}_m^{-1}) \times \ldots \times w(\overrightarrow{e}_1^{-1})}$$

for all $i \in \mathbb{Z}$. Therefore, it is recurrent if and only if the above ratio is equal to $1/2$, i.e. $w(\overrightarrow{e}_1) \times \ldots \times w(\overrightarrow{e}_m) = w(\overrightarrow{e}_m^{-1}) \times \ldots \times w(\overrightarrow{e}_1^{-1})$.

Assume now that **A.2** holds.
If **A.3** does not hold, then at least one edge of $\mathcal{G}$ has exactly one orientation $\overrightarrow{e}$ with positive weight. In this case, the RW on $(\mathcal{T}_\mathcal{G}, \circ)$ is not irreducible. However, due to Lemma 4.1.9 and **A.1**, a RW $(\mathcal{X}_t)_{t\geq 0}$ started at $\circ$ has probability at least $w_{min}^{|\overrightarrow{E}_\mathcal{G}|}$ to cross an oriented edge with label $\overrightarrow{e}$ after at most $|\overrightarrow{E}_\mathcal{G}|$ steps. Hence, if $T := \inf\{t \geq 0, \mathcal{X}_t \text{ has label } \overrightarrow{e}\}$, then T is stochastically dominated by a geometrical random variable of parameter $p \in (0,1)$ depending on $\mathcal{G}$. After this crossing, since $\mathcal{T}_\mathcal{G}$ is a tree, the RW can never come back to its starting point. Hence, it is transient.
Assume now that **A.3** holds, so that the RW on $(\mathcal{T}_\mathcal{G}, \circ)$ is irreducible (since all oriented edges have a positive weight). By Lemma 5.1 in [78], it is enough to prove Proposition 4.1.10 when the RW associated to $\mathcal{G}$ (and hence to $(\mathcal{T}_\mathcal{G}, \circ)$) is reversible. Indeed, this result states that if a discrete irreducible Markov chain admits an invariant measure, then it is transient if the additive reversibilization of the chain is. The additive reversibilization of the RW on $(\mathcal{T}_\mathcal{G}, \circ)$ is the RW on $(\mathcal{T}^*_\mathcal{G}, \circ)$, where $(\mathcal{T}^*_\mathcal{G}, \circ)$ is obtained from $(\mathcal{T}_\mathcal{G}, \circ)$ by modifying its weights as follows: for all $x, y \in V_{\mathcal{T}_\mathcal{G}}$, set

$$w^*(x,y) = \frac{1}{2}\left(w(x,y) + \frac{\tilde{\pi}(y)}{\tilde{\pi}(x)} w(y,x)\right).$$

$(\mathcal{T}^*_\mathcal{G}, \circ)$ is the universal cover of the graph $\mathcal{G}^*$ defined as follows. Define

$$w^*(\overrightarrow{e}) := \tfrac{1}{2}\left(w(\overrightarrow{e}) + \tfrac{\pi(u_2)}{\pi(u_1)} w(\overrightarrow{e}^{-1})\right)$$

for every $\overrightarrow{e} \in \overrightarrow{E}_\mathcal{G}$ with initial vertex u_1 and end vertex u_2. Let $\mathcal{G}^*$ be the weighted graph with the same vertex and edge sets as $\mathcal{G}$, and with edge weights $w^*(\overrightarrow{e})$ for every $\overrightarrow{e} \in \overrightarrow{E}_\mathcal{G}$. By construction of $\mathcal{G}^*$, the RW on $\mathcal{G}^*$ is reversible. Therefore, the RW on $(\mathcal{T}_\mathcal{G}, \circ)$ is transient if the RW on the universal cover of a reversible base graph $\mathcal{G}^*$ is transient.
Now, we can apply a classical transience criterion for discrete reversible Markov chains. The earliest proof we found traces back to 1983 (see [106]), and is derived from an analogous theorem of Royden for Riemannian surfaces. It states that if a discrete reversible Markov chain has state space Ω, transition matrix $\mathbf{P}$ and invariant measure π, and if there exists a collection of weights $\nu = (\nu_{i,j})_{i,j\in\Omega}$ such that:

(i) $\forall i, j \in \Omega$, $\nu_{i,j} = -\nu_{j,i}$,

(ii) there exists $i_0 \in \Omega$ such that for all $i \in \Omega$, $\sum_{j\in\Omega} \nu_{i,j} = \begin{cases} 1 \text{ if } i = i_0, \\ 0 \text{ else,} \end{cases}$

(iii) $\sum_{i,j\in\Omega} \frac{\nu_{i,j}^2}{\pi(i)\mathbf{P}(i,j)} < \infty$,

then the Markov chain is transient (here, we use the convention $0/0 = 0$). Looking at conditions (i) and (ii), we can interpret u as a current flow entering at i_0 and spreading to infinity through an electrical network. The condition (iii) states that the kinetic (or electric) energy of the flow is finite.

In our setting, we consider the flow generated by a symmetric NBRW starting at $\circ$, i.e. moving at each step to a uniform random child of its current position (hence not regarding the weights $w(\overrightarrow{e})$): for all $x \in (\mathcal{T}_{\mathcal{G}}, \circ)$ of height $R \geq 2$, let x_1 be x's parent and x_2 be x_1's parent, and let

$$\nu_{x_1,x} = \frac{\nu_{x_2,x_1}}{deg(x_1) - 1},$$

and set $\nu_{x,x_1} = -\nu_{x_1,x}$. For every child x of $\circ$, let $\nu_{\circ,x} = -\nu_{x,\circ} = 1/deg(\circ)$. For all $x, y \in V_{\mathcal{T}_{\mathcal{G}}}$ such that none is the parent of the other, let $\nu_{x,y} = 0$.
Then (i) holds obviously. (ii) is also straightforward. As for checking (iii), note that $\nu_{x_1,x}$ is the probability that a symmetric NBRW started at $\circ$ visits x. Hence the sum of the transitions from one generation to the next is always 1, that is, for all $R \geq 0$,

$$\sum_{x \in \partial B(\circ,R), y \in \partial B(\circ,R+1)} \nu_{x,y} = 1.$$

Remark that there exists $\varepsilon > 0$, only depending on $\mathcal{G}$, such that for all neighbours $x, y \in V_{\mathcal{T}_{\mathcal{G}}}$, $\widetilde{\pi}(x) P_{\mathcal{T}_{\mathcal{G}}}(x, y) > \varepsilon$. Therefore,

$$\begin{aligned}\sum_{x,y \in V_{\mathcal{T}_{\mathcal{G}}}} \frac{\nu_{x,y}^2}{\widetilde{\pi}(x) P_{\mathcal{T}_{\mathcal{G}}}(x,y)} &< 2\varepsilon^{-1} \sum_{R \geq 0} \left(\sum_{(x,y) \in \partial \overrightarrow{B}(\circ,R)} \nu_{x,y}^2 \right) \\ &\leq 2\varepsilon^{-1} \sum_{R \geq 0} \max_{(x,y) \in \partial \overrightarrow{B}(\circ,R)} \nu_{x,y},\end{aligned}$$

where $\partial \overrightarrow{B}(\circ, R)$ is the set of oriented edges with initial vertex in $\partial B(\circ, R)$ and end vertex in $\partial B(\circ, R+1)$. Hence to check (iii), it is enough to prove that the sequence $(s_R)_{R \geq 1}$ is summable with $s_R := \max_{(x,y) \in \partial \overrightarrow{B}(\circ,R)} \nu_{x,y}$.
One checks easily that the irreducibility of the RW on $\mathcal{G}$, **A.2** and **A.4** together imply that every non-backtracking path on $\mathcal{G}$, after at most $|\overrightarrow{E}_{\mathcal{G}}|$ steps, meets an oriented edge $\overrightarrow{e}$ leading to at least two other oriented edges. Hence for all $x \in (\mathcal{T}_{\mathcal{G}}, \circ)$, if $R = he(x)$ and x_1 is the parent of x, the shortest path from $\circ$ to x_1 contains at least $\lfloor R - 1/|\overrightarrow{E}_{\mathcal{G}}| \rfloor$ such edges $\overrightarrow{e}$, so that

$$\nu_{x_1,x} \leq K(1/2)^{R/|\overrightarrow{E}_{\mathcal{G}}|} \tag{4.7}$$

for some positive constant K. Hence $s_R \leq K(1 - w_{min})^{R/|\overrightarrow{E}_{\mathcal{G}}|}$ for all $R \geq 0$, so that $(s_R)_{R \geq 0}$ decays geometrically and the transience is proved. This concludes the proof. $\square$

Suppose now that **A.4** does not hold. $\mathcal{G}$ can be decomposed into a **core** $c(\mathcal{G})$ satisfying **A.4** and "branches" attached to it. $c(\mathcal{G})$ is constructed as follows: erase all vertices x of $\mathcal{G}$ such that $deg(x) = 1$ (call them **leaves**), and delete the edge attached to x. Perform this process again on the resulting multigraph, and so on, until no more vertex is erased. Denote $\mathcal{G}'$ the graph obtained by this algorithm. For every oriented edge $\overrightarrow{e}$ of $\mathcal{G}'$, change its weight to $w(\overrightarrow{e})/w'(x)$, where x is the initial vertex of $\overrightarrow{e}$ and $w'(x)$ is the total weight of the edges in $\mathcal{G}'$ starting at x: this modified graph is $c(\mathcal{G})$. Clearly, **A.4** holds on $c(\mathcal{G})$. Moreover, the RW associated to $c(\mathcal{G})$ is also irreducible (erasing a leaf from $\mathcal{G}$ does not affect the irreducibility).

Lemma 4.1.11. *The RW on $\mathcal{T}_{\mathcal{G}}$ is transient if and only if the RW on $\mathcal{T}_{c(\mathcal{G})}$ is.*

Proof. Remark that $c(\mathcal{T}_{\mathcal{G}}) = \mathcal{T}_{c(\mathcal{G})}$, the erased vertices in $\mathcal{T}_{\mathcal{G}}$ being exactly those whose labels are erased in $\mathcal{G}$. Let $(\mathcal{X}_t)_{t\geq 0}$ be a RW on $\mathcal{T}_{\mathcal{G}}$. Note that a.s., $(\mathcal{X}_t)_{t\geq 0}$ visits infinitely many distinct vertices and edges of $\mathcal{T}_{c(\mathcal{G})}$. The **trace of** $(\mathcal{X}_n)$ **on** $\mathcal{T}_{c(\mathcal{G})}$ is defined as follows: for all $p > m \geq 1$ such that $(\mathcal{X}_{m-1}, \mathcal{X}_m), (\mathcal{X}_p, \mathcal{X}_{p+1}) \in \mathcal{T}_{c(\mathcal{G})}$ and $(\mathcal{X}_i, \mathcal{X}_{i+1}) \notin \mathcal{T}_{c(\mathcal{G})}$ for all $m \leq i \leq p-1$, erase $\mathcal{X}_m, \mathcal{X}_{m+1}, \ldots, \mathcal{X}_{p-1}$. Denote $(\mathcal{X}'_t)_{t\geq 0}$ the sequence of remaining $\mathcal{X}_i$'s (ordered by increasing labels). Remark that the way the weights are chosen in the definition of $c(\mathcal{G})$ implies that $(\mathcal{X}'_t)$ is a RW on $\mathcal{T}_{c(\mathcal{G})}$, and it is transient if and only if $(\mathcal{X}_t)$ is. □

From now on and until the end of Section 4.1.4, assume that $\mathcal{G}$ satisfies **A.1**, **A.2**, **A.3** and **A.4**. Then Proposition 4.1.10 implies the following:

Corollary 4.1.12. *The RW associated to $(\mathcal{T}_{\mathcal{G}}, \circ)$ is transient.*

In Section 4.1.1, we defined the **ray to infinity** ξ of a RW $(\mathcal{X}_t)_{t\geq 0}$ in $(\mathcal{T}_{\mathcal{G}}, \circ)$ as:

$$\xi := \{x \in V_{\mathcal{T}_{\mathcal{G}}} \mid \exists s \geq 0, \mathcal{X}_s = x \text{ and } \forall t \geq s,\ \mathcal{X}_t \in V_x\},$$

with $\xi_t := \xi \cap \partial B(\mathcal{X}_0, t)$. The next Proposition expresses the following fact: due to the regularity of $\mathcal{T}_{\mathcal{G}}$, the RW is "uniformly transient" and the probability to make R steps in a given direction decreases exponentially w.r.t. R.

Proposition 4.1.13. *For every base graph $\mathcal{G}$, there exist two positive constants C_1, C_2 so that if $(\mathcal{X}_t)_{t\geq 0}$ is a RW on $(\mathcal{T}_{\mathcal{G}}, \circ)$, then for all $R \geq 0$ and $x \notin B(\circ, R)$,*

$$\mathbb{P}(\exists t > 0,\ X_t = x) \leq C_1 \exp(-C_2 R).$$

Since those constants are independent of the choice of $\circ$, the Markov Property gives the following generalization: for all $s > 0$, $y \in (\mathcal{T}_{\mathcal{G}}, \circ)$ and $x \notin B(y, R)$,

$$\mathbb{P}(\exists t > s,\ X_t = x | X_s = y) \leq C_1 \exp(-C_2 R).$$

The proof of this proposition requires the following intermediate result.

Lemma 4.1.14. *There exists a positive constant C_0, independent of the choice of v_*, such that for every child x of $\circ$, if $(\mathcal{X}_t)_{t\geq 0}$ is a RW starting at $\circ$, then:*

$$\mathbb{P}(\forall t \geq 1,\ \mathcal{X}_t \in V_x) \geq C_0.$$

Proof. Since the RW on $\mathcal{T}_\mathcal{G}$ is transient, there exists an oriented edge $(x_1, x_2) \in \overrightarrow{E}_{\mathcal{T}_\mathcal{G}}$ such that a RW started at x_1 has a probability $p > 0$ to visit x_2 at its first step and to never come back to x_1. Note that this holds with the same value of p if one replaces x_1 (resp. x_2) by any $x'_1 \in V_{\mathcal{T}_\mathcal{G}}$ (resp. any $x'_2 \in V_{\mathcal{T}_\mathcal{G}}$) such that (x'_1, x'_2) has the same label as (x_1, x_2). Let $\overrightarrow{e}$ be this label.
We claim that **A.1**, **A.2** and **A.4** together imply that:

$$\text{for any } \overrightarrow{e}_a, \overrightarrow{e}_b \in \overrightarrow{E}_\mathcal{G} \text{ with positive weight, there exists a non-backtracking path } (\overrightarrow{e}_1, \dots, \overrightarrow{e}_m) \text{ in } \mathcal{G} \text{ with } m \leq |\overrightarrow{E}_\mathcal{G}| \text{ such that } \overrightarrow{e}_1 = \overrightarrow{e}_a \text{ and } \overrightarrow{e}_m = \overrightarrow{e}_b. \tag{4.8}$$

We apply this with the label of $(\circ, x)$ as $\overrightarrow{e}_a$ and $\overrightarrow{e}_b = \overrightarrow{e}$. Together with the projection property of $(\mathcal{T}_\mathcal{G}, \circ)$ on $\mathcal{G}$ (Lemma 4.1.9), it implies that there is an upward path $(\overrightarrow{e}_1, \dots, \overrightarrow{e}_m)$ in $\mathcal{T}_\mathcal{G}$ such that $m \leq |\overrightarrow{E}_\mathcal{G}|$, $\overrightarrow{e}_1 = (\circ, x)$ and $\overrightarrow{e}_m$ has label $\overrightarrow{e}$ (recall the definition of upward path from the end of Section 4.1.3). Note that we have $\{(\mathcal{X}_t) \text{ follows } (\overrightarrow{e}_1, \dots, \overrightarrow{e}_m) \text{ and then remains in } \mathcal{T}_{\mathcal{X}_m}\} \subseteq \{\forall t \geq 1, \mathcal{X}_t \in V_x\}$. Hence the Markov Property of the RW on $\mathcal{T}_\mathcal{G}$ implies that

$$\mathbb{P}(\forall t \geq 1, \mathcal{X}_t \in V_x) \geq (w_{min})^m \times p \geq w_{min}^{|\overrightarrow{E}_\mathcal{G}|} p,$$

and we can take $C_0 = w_{min}^{|\overrightarrow{E}_\mathcal{G}|} p$.
It remains to prove (4.8). Let $\overrightarrow{e}_a, \overrightarrow{e}_b \in \overrightarrow{E}_\mathcal{G}$. The irreducibility of the RW (**A.1**) implies that there is a non-backtracking path starting from $\overrightarrow{e}_a$ and leading to an oriented edge whose end vertex is the initial vertex x_b of $\overrightarrow{e}_b$. Hence, the NBRW can reach $\overrightarrow{e}_b$ or $\overrightarrow{e}_b^{-1}$ (note that if the NBRW reaches x_b through $\overrightarrow{e}_b{-1}$, its next step cannot be $\overrightarrow{e}_b$). If $w(\overrightarrow{e}_b^{-1}) = 0$, then the NBRW might arrive at x_b through another edge than $\overrightarrow{e}_b^{-1}$, hence its next step can be $\overrightarrow{e}_b$.
It remains to show that if $w(\overrightarrow{e}_b)w(\overrightarrow{e}_b^{-1}) \neq 0$, there is a non-backtracking path from $\overrightarrow{e}_b^{-1}$ to $\overrightarrow{e}_b$. By **A.4**, there is a cycle $\mathcal{C}$ through $\overrightarrow{e}_b$ and a cycle $\mathcal{C}'$ through $\overrightarrow{e}_b^{-1}$, and we can impose by **A.2** that none of them is reduced to $\{\overrightarrow{e}_b, \overrightarrow{e}_b^{-1}\}$.
If $\mathcal{C}'$ is not the inverse of $\mathcal{C}$, let $\overrightarrow{e}_c$ be the first oriented edge of $\mathcal{C}$ (starting from $\overrightarrow{e}_b$) such that $\overrightarrow{e}_c^{-1}$ is not in $\mathcal{C}'$, and x_c its initial vertex. Then x_c is the end vertex of an oriented edge $\overrightarrow{e}_d$ of $\mathcal{C}'$. Therefore, a NBRW can go from $\overrightarrow{e}_b^{-1}$ to $\overrightarrow{e}_d$, then to $\overrightarrow{e}_c$ and finally to $\overrightarrow{e}_b$ since $\overrightarrow{e}_c, \overrightarrow{e}_b \in \mathcal{C}$.
If $\mathcal{C}'$ is the inverse of $\mathcal{C}$, by **A.2**, there is another oriented cycle in $\mathcal{G}$. Since $\mathcal{G}$ is connected, for some vertex x_c of $\mathcal{C}$, there is an oriented edge $\overrightarrow{e}_c \in \overrightarrow{E}_\mathcal{G} \setminus \{\mathcal{C} \cup \mathcal{C}'\}$, with initial vertex x_c. By **A.1**, there exists a non-backtracking path starting at $\overrightarrow{e}_c$ and ending at some oriented edge $\overrightarrow{e}_d \in \overrightarrow{E}_\mathcal{G} \setminus \{\mathcal{C} \cup \mathcal{C}'\}$ whose end vertex x_d is on $\mathcal{C}$. Note that there is a non-backtracking path from $\overrightarrow{e}_d$ to any oriented edge of $\mathcal{C}$. Hence, there is a non-backtracking path from $\overrightarrow{e}_b^{-1}$ to $\overrightarrow{e}_c$, then to $\overrightarrow{e}_d$ and finally to $\overrightarrow{e}_b$.
Moreover, one might impose that this path from $\overrightarrow{e}_a$ to $\overrightarrow{e}_b$ is injective (by deleting its cycles), so that $m \leq |\overrightarrow{E}_\mathcal{G}|$. ☐

Proof of Proposition 4.1.13. Let $R \geq 0$ and $x \in V_{\mathcal{T}_\mathcal{G}}$. By the same reasoning as in the paragraph above (4.7), on the shortest path p from $\circ$ to x, there are at least $m := \lfloor (R-2)/|\overrightarrow{E}_\mathcal{G}| \rfloor$ vertices (excluding $\circ$ and x) that are the initial vertex of two upward edges in $(\mathcal{T}_{\mathcal{G}}, \circ)$. Denote $x_1, \dots x_m$ the first m such vertices by increasing height, and y_i the child of x_i that is on p. Let E_i (resp. E'_i) be the event that $(\mathcal{X}_t)_{t\geq 0}$ hits x_i (resp. y_i). By Lemma 4.1.14, after hitting x_i, the RW has a probability at least C_0 to escape through the child of x_i that is not on p, and to never hit y_i. Hence by the strong Markov property, $\mathbb{P}(E'_i|E_i) \leq 1 - C_0$. Note that $E'_m \subseteq E_m \subseteq E'_{m-1} \subseteq \ldots \subseteq E'_1 \subseteq E_1$. Therefore,

$$\begin{aligned}\mathbb{P}(\exists t > 0,\, \mathcal{X}_t = x) \leq & \mathbb{P}(E'_m) \\ \leq & \mathbb{P}(\cap_{i=1}^m (E_i \cap E'_i)) \\ \leq & \mathbb{P}(E_1) \times \prod_{i=1}^{m-1} \mathbb{P}(E'_i|E_i)\mathbb{P}(E_{i+1}|E'_i) \times \mathbb{P}(E'_m|E_m) \\ \leq & (1 - C_0)^m.\end{aligned}$$

The conclusion follows. □

As a corollary, we can prove Proposition 4.1.4:

Proof of Proposition 4.1.4. Note that $\xi \cap B(\mathcal{X}_t, R) = \emptyset$ implies that there exists $y \notin B(\mathcal{X}_t, R)$ such that for some $s > t$, $\mathcal{X}_s = y$, so that we can apply the previous Proposition. □

CLTs for the RW on the universal cover

For $x, y \in V_{\mathcal{T}_{\mathcal{G}}}$ such that $w(x, y) > 0$, let $\hat{w}(x, y) := \mathbb{P}(y \in \xi | \mathcal{X}_0 = x)$ be the probability that the ray to infinity of a RW $(\mathcal{X}_t)_{t\geq 0}$ started at x goes through a given neighbour y of x. Note that this quantity only depends on the label of (x, y) (denote it $\overrightarrow{e}$), so that one might define $\hat{w}(\overrightarrow{e}) = \hat{w}(x, y)$. Let $\widehat{\mathcal{G}}$ be $\mathcal{G}$ with weights $(\hat{w}(\overrightarrow{e}))_{\overrightarrow{e} \in \overrightarrow{E}_{\mathcal{G}}}$ instead of $(w(\overrightarrow{e}))_{\overrightarrow{e} \in \overrightarrow{E}_{\mathcal{G}}}$.
Define the **Non Backtracking Random Walk (NBRW) on** $\widehat{\mathcal{G}}$ as a RW $(Z_t)_{t\geq 0}$ on $\overrightarrow{E}_{\mathcal{G}}$, such that for all $\overrightarrow{e}_1, \overrightarrow{e}_2 \in \overrightarrow{E}_{\mathcal{G}}$ and $t \geq 0$,

$$\mathbb{P}(Z_{t+1} = \overrightarrow{e}_2 | Z_t = \overrightarrow{e}_1) = \begin{cases} \frac{\hat{w}(\overrightarrow{e}_2)}{1 - \hat{w}(\overrightarrow{e}_1^{-1})} \text{ if the end vertex of } \overrightarrow{e}_1 \text{ is the initial} \\ \qquad\qquad \text{vertex of } \overrightarrow{e}_2 \text{ and } \overrightarrow{e}_2 \neq \overrightarrow{e}_1^{-1}, \\ 0 \text{ else.} \end{cases} \tag{4.9}$$

Lemma 4.1.14 ensures that for all $\overrightarrow{e} \in \overrightarrow{E}_{\mathcal{G}}$, $\hat{w}(\overrightarrow{e}) > 0$ if $w(\overrightarrow{e}) > 0$. In particular, by our assumptions on $\mathcal{G}$, and the claim in the proof of Lemma 4.1.14, the NBRW associated to $\widehat{\mathcal{G}}$ is irreducible, so that it has a unique invariant probability measure $\hat{\pi}$.

Proposition 4.1.15. (Theorem F' in [117]) *Let $(\mathcal{X}_t)$ be a RW on $\mathcal{T}_{\mathcal{G}}$ with $\mathcal{X}_0 = \circ$, and let $\rho_t := (\xi_t, \xi_{t+1})$ be the t-th upward edge of its ray to infinity ξ. Then $(\rho_t)_{t\geq 0}$ is a NBRW on $\mathcal{T}_{\widehat{\mathcal{G}}}$. Its first oriented edge is chosen among those with initial vertex $\circ$, with probability given by $\hat{w}$.*

Proof. The assertion on the choice of the first oriented edge of (ρ_t) is clear by definition of $\hat{w}$. It is then enough to prove that for all $t_0 \geq 1$, for all $x \in V_{\mathcal{T}_\mathcal{G}}$ of height t_0+1, and x_i the vertex of height i in the shortest path from $\circ$ to x, $i \in \{t_0-1, t_0\}$:

$$\mathbb{P}(\rho_{t_0+1} = (x_{t_0}, x) | \, \rho_{t_0} = (x_{t_0-1}, x_{t_0})) = \frac{\hat{w}(x_{t_0}, x)}{1-\hat{w}(x_{t_0}, x_{t_0-1})}. \tag{4.10}$$

For simplicity, write $p := \mathbb{P}(\rho_{t_0+1} = (x_{t_0}, x) | \, \rho_{t_0} = (x_{t_0-1}, x_{t_0}))$. Let $\tau = \inf\{t \geq 0, \, X_t = x_{t_0}\}$ and denote $(\rho_t^{(\tau)})_{t\geq 0}$ the ray to infinity of $(\mathcal{X}_{\tau+t})_{t\geq 0}$. We have $\{\rho_{t_0} = (x_{t_0-1}, x_{t_0})\} = \{\tau < +\infty\} \cap \{\rho_1^{(\tau)} \neq x_{t_0-1}\}$ and $\{\rho_{t_0+1} = (x_{t_0}, x)\} = \{\tau < +\infty\} \cap \{\rho_1^{(\tau)} = x\}$. Let $(\rho_t^0)_{t\geq 0}$ be the ray to infinity of a RW started at x_{t_0}. Then

$$\begin{aligned} p &= \frac{\mathbb{P}(\{\tau < +\infty\} \cap \{\rho_1^{(\tau)} = x\})}{\mathbb{P}(\{\tau < +\infty\} \cap \{\rho_1^{(\tau)} \neq x_{t_0-1}\})} \\ &= \frac{\mathbb{P}(\tau < +\infty)\mathbb{P}(\rho_1^0 = x)}{\mathbb{P}(\tau < +\infty)\mathbb{P}(\rho_1^0 \neq x_{t_0-1})} \\ &= \frac{\hat{w}(x_{t_0}, x)}{1-\hat{w}(x_{t_0}, x_{t_0-1})} \end{aligned}$$

by the Strong Markov Property applied to the stopping time τ. □

We define the **entropic weight of a vertex** $x \in V_{\mathcal{T}_\mathcal{G}}$ as the probability that x is in the ray to infinity of a RW started at $\circ$, and denote it $W(x)$. Let $(x_0, \ldots, x_H)$ be the vertices on the shortest path from $\circ$ to x, where $H = he(x)$, so that $x_0 = \circ$ and $x_H = x$. By Proposition 4.1.15,

$$W(x) = \hat{w}(x_0, x_1) \times \prod_{i=1}^{H-1} \frac{\hat{w}(x_i, x_{i+1})}{1-\hat{w}(x_i, x_{i-1})}. \tag{4.11}$$

Remark that the transition probabilities for the first step of the ray to infinity are different, since one does not condition on the event that the RW starting at x_i will not go to infinity through x_{i-1}.

Corollary 4.1.16. *There exist $h_W > 0$, $\sigma_W \geq 0$ such that for all $x \in V_{\mathcal{T}_\mathcal{G}}$ and $(\mathcal{X}_t)_{t\geq 0}$ a RW on $(\mathcal{T}_\mathcal{G}, \circ)$ with $\mathcal{X}_0 = \circ$,*

$$\frac{-\log W(\xi_t) - h_W^{-1} t}{\sqrt{t}} \underset{t\to+\infty}{\to} \mathcal{N}(0, \sigma_W^2)$$

in distribution.

Proof. From equation (4.11), we have $W(\xi_t) = \frac{\prod_{i=0}^{t-1} \hat{w}(\ell(\rho_i))}{\prod_{i=0}^{t-2} 1-\hat{w}(\ell(\rho_i))}$, so that

$$-\log W(\xi_t) = \log \hat{w}(\ell(\rho_{t-1})) + \sum_{i=0}^{t-2} \left[\log \hat{w}(\ell(\rho_i)) - \log(1-\hat{w}(\ell(\rho_i)))\right],$$

where $\ell(\overrightarrow{e})$ is the label of $\overrightarrow{e}$ for every oriented edge $\overrightarrow{e}$. By Proposition 4.1.15 and Lemma 4.1.9, $(\ell(\rho_i))_{i\geq 0}$ is a NBRW on $\widehat{\mathcal{G}}$. We conclude by Lemma 4.1.8, where the Markov chain is the NBRW on $\widehat{\mathcal{G}}$ (hence $\Omega = \overrightarrow{E}_\mathcal{G}$), and with $f(\overrightarrow{e}) = \log \hat{w}(\overrightarrow{e}) - \log(1-\hat{w}(\overrightarrow{e}))$. □

It remains now to derive a similar result for $(\log W(\mathcal{X}_t))_{t\geq 0}$, which requires in particular to prove that the RW has a positive speed. Since $(\mathcal{X}_t)_{t\geq 0}$ is transient, $\theta_i := \sup\{t \geq 0 \,|\, he(\mathcal{X}_t) = i\}$ is a.s. well-defined and finite for all $i \in \mathbb{N}$. Let $\tilde{\theta}_i := \theta_{i+1} - \theta_i$ be the time between the last visits of the walk at level i and at level $i+1$, for $i \geq 1$. Let $\tilde{\theta}_0 := \theta_1$. Note that $\xi_i = \mathcal{X}_{\theta_i}$. We have the following corollary of Theorem 2.5 in [94] (or Theorem C in [117]):

Proposition 4.1.17. *There exist $C_3, C_4 > 0$ such that for all $n \geq 0$ and $x \in (\mathcal{T}_{\mathcal{G}}, \circ)$,*

$$P^n_{\mathcal{T}_{\mathcal{G}}}(x,x) \leq C_3 \exp(-C_4 n). \tag{4.12}$$

We deduce the following:

Corollary 4.1.18. *There exists $C_5 > 0$ such that for all $i, n \geq 0$, and for all $\overrightarrow{e} \in \overrightarrow{E}_{\mathcal{G}}$ such that $\mathbb{P}((\mathcal{X}_{\theta_i - 1}, \mathcal{X}_{\theta_i})$ has label $\overrightarrow{e}) > 0$,*

$$\mathbb{P}(\tilde{\theta}_i \geq n |\, (\mathcal{X}_{\theta_i-1}, X_{\theta_i}) \text{ has label } \overrightarrow{e}) \leq C_5 \exp(-C_4 n) \tag{4.13}$$

when $i > 0$ and $\mathbb{P}(\tilde{\theta}_0 \geq n) \leq C_5 \exp(-C_4 n)$.

Proof. For $i = 0$, this is a direct application of the above Proposition 4.1.17. Now, for $i \geq 1$, note that the law of $(\mathcal{X}_t)_{t\geq\theta_i}$ is that of a RW started at $\mathcal{X}_{\theta_i}$ conditioned on making its first step not towards the parent of $\mathcal{X}_{\theta_i}$, and then never coming back to $\mathcal{X}_{\theta_i}$ after the start. If A denotes the event that this conditioning happens, then $\mathbb{P}(A) \geq w_{min} C_0$ by Lemma 4.1.14. Therefore, (4.12) implies that for all $n \geq 1$,

$$\begin{aligned}
\mathbb{P}(\tilde{\theta}_i \geq n) \leq & \sum_{m\geq n} \mathbb{P}(X_{\theta_i + m} = X_{\theta_i+1}) \\
\leq & \frac{1}{w_{min} C_0} \sum_{m \geq n} C_3 \exp(-C_4 m) \\
\leq & C_5 \exp(-C_4 n)
\end{aligned}$$

for $C_5 \geq \frac{C_3}{w_{min} C_0}$. □

We say that $t \in \mathbb{N}$ is an **exit time** if the oriented edge $(\mathcal{X}_t, \mathcal{X}_{t+1})$ is upward, has label $\overrightarrow{e}_*$ and if $\mathcal{X}_s \neq \mathcal{X}_t$ for all $s \geq t+1$ (recall that $\overrightarrow{e}_* \in \overrightarrow{E}_{\mathcal{G}}$ was arbitrarily picked at the beginning of Section 4.1.3). For $i \geq 1$, let τ_i be the i-th exit time and $\overrightarrow{e}_i$ the corresponding **exit edge**. Note that the $\overrightarrow{e}_i$'s are exactly the edges of type $\overrightarrow{e}$ in ρ. Let $\tau_0 := 0$ and $\widetilde{\tau}_i := \tau_{i+1} - \tau_i$ for $i \geq 0$ be the i**-th renewal interval**. Remark that Proposition 4.1.15 implies that $\tau_i < \infty$ for all $i \geq 1$ a.s.

Let $\epsilon_i := (\mathcal{X}_t)_{\tau_i+1 \leq t \leq \tau_{i+1}}$ be the i-th **excursion** between two exit times. Let x_* be the end vertex of the oriented edge of label $\overrightarrow{e}_*$ starting at $\circ$. Let $\bar{\epsilon}_i$ be the projection of ϵ_i on $(\mathcal{T}_{x_*}, x_*)$, the subtree of $(\mathcal{T}_{\mathcal{G}}, \circ)$ rooted at x_*: for every $i \geq 1$, there is an isomorphism ϕ_i from $(\mathcal{T}_{\mathcal{X}_{\tau_i+1}}, \mathcal{X}_{\tau_i+1})$ to $(\mathcal{T}_{x_*}, x_*)$ since $\mathcal{X}_{\tau_i+1}$ and x_* have the same label. Let $\overline{\mathcal{X}}_t := \phi_i(\mathcal{X}_t)$ for $t \geq \tau_i + 1$ and define $\bar{\epsilon}_i := (\overline{\mathcal{X}}_t)_{\tau_i+1 \leq t \leq \tau_{i+1}}$.

Proposition 4.1.19. *The random variables $(\overline{\epsilon}_i)_{i\geq 1}$ are i.i.d., and there exist $C_6, C_7 > 0$ such that for all $m, i \geq 0$,*

$$\mathbb{P}(\widetilde{\tau}_i \geq m) \leq C_6 \exp(-C_7 m). \tag{4.14}$$

Proof. Note that for all $i \geq 1$,

$$(\overline{\mathcal{X}}_{\tau_i+t})_{t\geq 1} \overset{law}{=} (\widetilde{\mathcal{X}}_t)_{t\geq 0}, \tag{4.15}$$

where $(\widetilde{\mathcal{X}}_t)$ is a RW on $(\mathcal{T}_\mathcal{G}, \circ)$ starting at x_* and conditioned on never visiting $\circ$. Hence the variables $\overline{\epsilon}_i$ all have the same distribution.

As for the independence, note that for all $i \geq 1$, conditionally on $\tau_1, \ldots, \tau_i$ and $\mathcal{X}_0, \ldots, \mathcal{X}_{\tau_i}$, (4.15) holds. In particular, $\overline{\epsilon}_i$ is independent from the sigma-algebra $\sigma(\epsilon_0, \ldots, \epsilon_{i-1})$. From this, we deduce that the $\overline{\epsilon}_i$'s are together independent.

We now prove (4.14). Remark that conditionally on the label of the oriented edge $(\mathcal{X}_{\theta_i-1}, \mathcal{X}_{\theta_i})$, $\tilde{\theta}_i$ is independent of $\sigma\left((\tilde{\theta}_k)_{0\leq k\leq i-1}\right)$. By Corollary 4.1.18, there exists a probability distribution Θ on $\mathbb{N}$ with expectation $\mathcal{E} := \mathbb{E}[\Theta] < +\infty$ such that $\mathbb{P}(\Theta \geq n) \leq C_5 \exp(-C_4 n)$ for all $n \geq 1$ and such that θ_i is stochastically dominated by $Z_1 + \ldots + Z_i$, where the Z_j's are i.i.d. variables of law Θ.

For all $t > 0$ and all $n \geq 1$,

$$\begin{aligned}
\mathbb{P}(\theta_{\lfloor n/2\mathcal{E}\rfloor} \geq n) \leq& \mathbb{P}(Z_1 + \ldots + Z_{\lfloor n/2\mathcal{E}\rfloor} \geq n) \\
\leq& \frac{\mathbb{E}[\exp(tZ_1)]^{\lfloor n/2\mathcal{E}\rfloor}}{e^{tn}} \\
\leq& \left(\frac{\mathbb{E}[\exp(tZ_1)]}{e^{2\mathcal{E}t}}\right)^{\lfloor n/2\mathcal{E}\rfloor}
\end{aligned}$$

for all $t > 0$ by Markov's inequality. Note that for $t < C_4$, $\mathbb{E}[\exp(tZ_1)]$ is finite, and that for $t \to 0$, $\mathbb{E}[\exp(tZ_1)] = \sum_{j\geq 0} t^j \frac{\mathbb{E}[Z_1^j]}{j!} = 1 + \mathcal{E}t + O(t^2)$. Thus, there exists $t_0 > 0$ such that $0 < r < 1$ where $r := \mathbb{E}[\exp(t_0 Z_1)] \exp(-2\mathcal{E}t_0)$, and we have $\mathbb{P}(\theta_{\lfloor n/2\mathcal{E}\rfloor} \geq n) \leq r^{\lfloor n/2\mathcal{E}\rfloor}$.

Let $\vartheta := \inf\{k \geq 1, \rho_k \text{ has type } \overrightarrow{e}\}$. Remark that $\hat{w}(\overrightarrow{e}') > 0$ if and only if $\hat{w}(e') > 0$, for every oriented edge $\overrightarrow{e}'$. Hence (4.8) holds on $\overrightarrow{E}_\mathcal{G}$, and there exists $p > 0$ such that for any $\overrightarrow{e}' \in \overrightarrow{E}_\mathcal{G}$, a NBRW on $\overrightarrow{E}_\mathcal{G}$ starting at $\overrightarrow{e}'$ has a probability at least p to visit $\overrightarrow{e}$ after at most $|\overrightarrow{E}_\mathcal{G}|$ steps. By Proposition 4.1.15 and Lemma 4.1.9, ϑ is stochastically dominated by $|\overrightarrow{E}_\mathcal{G}|$ times a geometrical variable whose parameter only depends on $\mathcal{G}$ (recall that the NBRW on $\widehat{\mathcal{G}}$ is irreducible, so that a NBRW on $\widehat{\mathcal{G}}$ will go through $\overrightarrow{e}$ after a geometrical time, independently of its starting position).

Thus, there exist constants $\alpha, \beta > 0$ such that for all $m \geq 0$, $\mathbb{P}(\overrightarrow{e}_1, \overrightarrow{e}_2 \notin \{\rho_1, \ldots, \rho_m\}) \leq \alpha \exp(-\beta m)$, implying that

$$\mathbb{P}(\tau_2 \geq \theta_m) \leq \alpha \exp(-\beta m).$$

Hence, for all $i \geq 0$, since the $(\widetilde{\tau}_i)_{i\geq 1}$ are i.i.d.,

$$\begin{aligned}\mathbb{P}(\widetilde{\tau}_i \geq n) &\leq \mathbb{P}(\widetilde{\tau}_0 + \widetilde{\tau}_1 \geq n)\\ &\leq \mathbb{P}(\theta_{\lfloor n/2\mathcal{E}\rfloor} \geq n) + \mathbb{P}(\tau_2 \geq \theta_{\lfloor n/2\mathcal{E}\rfloor})\\ &\leq r^{\lfloor n/2\mathcal{E}\rfloor} + \alpha \exp(-\beta\lfloor n/2\mathcal{E}\rfloor).\end{aligned}$$

This conclude the proof. □

Recall that $W_t := -\log W(\mathcal{X}_t)$ is the log-weight of the RW. We now prove Theorem 4.1.3.

Proof of Theorem 4.1.3. Let $W_i^{(\epsilon)} := \log W(\mathcal{X}_{\tau_i}) - \log W(\mathcal{X}_{\tau_{i+1}})$ for every $i \geq 1$, and define the quantity $\hat{w}_{min} := \min_{e\in\overrightarrow{E}_{\mathcal{G}},\hat{w}(\overrightarrow{e})>0} \hat{w}(\overrightarrow{e})$. Remark that for every $i \geq 1$, $W_i^{(\epsilon)} \leq -\widetilde{\tau}_i \log \hat{w}_{min}$.
By Proposition 4.1.19, the $(W_i^{(\epsilon)})_{i\geq 1}$'s are i.i.d., and for some constant $C' > 0$ and all $n \geq 1$, $\mathbb{P}(W_1^{(\epsilon)} \geq n) \leq \mathbb{P}(\widetilde{\tau}_1 \geq -n/\log \hat{w}_{min}) = O(\exp(C'n/\log \hat{w}_{min}))$, so that $W_1^{(\epsilon)}$ has moments of any order. Let $h_w := \mathbb{E}[W_1^{(\epsilon)}]$. Clearly, $W_1^{(\epsilon)} > 0$ a.s., so that $h_w > 0$.
Now, cutting the trajectory of the RW into excursions between exit edges, we have

$$W_t = -\log W(\mathcal{X}_{\tau_1}) - \log W(\mathcal{X}_t) + \log W(\mathcal{X}_{\tau_{r_t}}) + \sum_{i=1}^{r_t-1} W_i^{(\epsilon)},$$

where $r_t := \max\{i \geq 0,\, \tau_i \leq t\}$ for all $t \geq 0$.
By Proposition 4.1.19 again, $\log W(\mathcal{X}_{\tau_1}) + \log W(\mathcal{X}_t) - \log W(\mathcal{X}_{\tau_{r_t}}) = o(\sqrt{t})$ with high probability, so that by Slutsky's Lemma, it is enough to show the existence of $h, \sigma > 0$ such that

$$\frac{\sum_{i=1}^{r_t-1} W_i^{(\epsilon)} - ht}{\sqrt{t}} \overset{law}{\to} \mathcal{N}(0, \sigma^2).$$

Let $\tau := \mathbb{E}[\widetilde{\tau}_1] \in (0, \infty)$, $\overline{W^{(\epsilon)}}_i := W_i^{(\epsilon)} - h_w$ and $\overline{\tau}_i := \widetilde{\tau}_i - \tau$. For all $\lambda \in \mathbb{R}$, if we set $p_\lambda := \mathbb{P}\left(\frac{\sum_{i=1}^{r_t-1} W_i^{(\epsilon)} - \frac{h_w t}{\tau}}{\sqrt{t}} \leq \lambda\right)$, we have

$$\begin{aligned}p_\lambda =& \mathbb{P}\left(\sum_{i=1}^{r_t-1} W_i^{(\epsilon)} \leq \frac{h_w t}{\tau} + \lambda\sqrt{t}\right)\\ =& \mathbb{P}\left(\frac{h_w r_t + \sum_{i=1}^{r_t-1}\overline{W^{(\epsilon)}}_i}{\tau r_t + \sum_{i=1}^{r_t-1}\overline{\tau}_i} \times \frac{\sum_{i=1}^{r_t-1}\widetilde{\tau}_i}{t} \leq \frac{h_w}{\tau} + \frac{\lambda}{\sqrt{t}}\right)\\ =& \mathbb{P}\left(\frac{h_w}{\tau}\left(\frac{1 + \sum_{i=1}^{r_t-1}\overline{W^{(\epsilon)}}_i/(h_w r_t)}{1 + \sum_{i=1}^{r_t-1}\overline{\tau}_i/(\tau r_t)}\right)\frac{\sum_{i=1}^{r_t-1}\widetilde{\tau}_i}{t} \leq \frac{h_w}{\tau} + \frac{\lambda}{\sqrt{t}}\right).\end{aligned}$$

The sequence $(W_i^{(\epsilon)})_{i\geq 1}$ is i.i.d., and $\mathbb{E}[\overline{W^{(\epsilon)}}_1] = 0$. Since $r_t \leq t$ a.s., we have by Doob's maximal inequality:

$$\mathbb{P}(|\textstyle\sum_{i=1}^{r_t-1}\overline{W^{(\epsilon)}}_i| \geq t^{3/5}) \leq \mathbb{P}(\sup_{1\leq j\leq t}|\sum_{i=1}^{j}\overline{W^{(\epsilon)}}_i| \geq t^{3/5}) = o(1).$$

Therefore, $\sum_{i=1}^{r_t-1} \overline{W^{(\epsilon)}}_i = o(t^{2/3})$ with high probability as $t \to +\infty$, and $\sum_{i=1}^{r_t-1} \overline{\tau}_i = o(t^{2/3})$ for the same reason. In addition, the strong law of large numbers implies that $r_t/t \to 1/\tau$ a.s. Hence,

$$\frac{1+\sum_{i=1}^{r_t-1} \overline{W^{(\epsilon)}}_i/(h_w r_t)}{1+\sum_{i=1}^{r_t-1} \overline{\tau}_i/(\tau r_t)} = 1 + \frac{1}{r_t}\sum_{i=1}^{r_t-1}\left(\frac{\overline{W^{(\epsilon)}}_i}{h_w} - \frac{\overline{\tau}_i}{\tau}\right) + R_t, \tag{4.16}$$

where $R_t = o(t^{-2/3})$ in probability.
Furthermore, $\frac{\sum_{i=1}^{r_t-1} \widetilde{\tau}_i}{t} = 1 + \frac{(\tau_{r_t}-t)-\tau_1}{t} = 1 + o(t^{-2/3})$ in probability, according to Proposition 4.1.19. Hence

$$\frac{h_w}{\tau}\left(\frac{1+\sum_{i=1}^{r_t-1} \overline{W^{(\epsilon)}}_i/(h_w r_t)}{1+\sum_{i=1}^{r_t-1} \overline{\tau}_i/(\tau r_t)}\right)\frac{\sum_{i=1}^{r_t-1}\widetilde{\tau}_i}{t} = \frac{h_w}{\tau} + \frac{h_w}{\tau r_t}\sum_{i=1}^{r_t-1} Z_i + R'_t,$$

where $Z_i := \frac{\overline{W^{(\epsilon)}}_i}{h_w} - \frac{\overline{\tau}_i}{\tau}$ and $R'_t = o(t^{-2/3})$ in probability, and

$$\begin{aligned}
\mathbb{P}\left(\frac{\sum_{i=1}^{r_t-1} W_i^{(\epsilon)} - \frac{h_w t}{\tau}}{\sqrt{t}} \le \lambda\right) =& \mathbb{P}\left(\frac{1}{r_t}\sum_{i=1}^{r_t-1} Z_i \le \frac{\tau\lambda}{h_w\sqrt{t}} - R'_t\right)\\
=& \mathbb{P}\left(\frac{1}{\sqrt{r_t}}\sum_{i=1}^{r_t-1} Z_i \le \frac{\tau\lambda}{h_w}\sqrt{\frac{r_t}{t}} - R'_t\sqrt{r_t}\right)\\
=& \mathbb{P}\left(\frac{1}{\sqrt{r_t}}\sum_{i=1}^{r_t-1} Z_i \le \frac{\sqrt{\tau}\lambda}{h_w} + R''_t\right),
\end{aligned}$$

where $R''_t = o(1)$ in probability.
The Z_i's are i.i.d. variables, and we have $\mathbb{E}[Z_1] = 0$ and $\sigma_Z^2 := \mathrm{Var}(Z_1) > 0$. Indeed by **A.3**, for a fixed trajectory of ε_1 (hence a given value of $\overline{W^{(\epsilon)}}_1$), $\overline{\tau}_1$ can take different values with positive probability, so that Z_1 is not deterministic. And Z_1 has exponential moments by Proposition 4.1.19, so that σ_Z^2 is finite.
Let $r'_t := \lfloor t/\tau \rfloor$. A consequence of the CLT applied to the series associated to the sequence $(\tau_i)_{i\ge 1}$ is that $|r_t - r'_t| \le t^{2/3}$ w.h.p., so that

$$\left|\sum_{i=1}^{r_t-1} Z_i - \sum_{i=1}^{r'_t-1} Z_i\right| \le (t^{2/3})^{3/5} \le t^{2/5} \text{ and } \left|\sum_{i=1}^{r_t-1} Z_i\right| \le t^{3/5}$$

w.h.p. by the same argument using Doob's maximal inequality as above (4.16). Moreover, we have w.h.p. $\left|\frac{1}{\sqrt{r_t}} - \frac{1}{\sqrt{r'_t}}\right| \le \frac{1}{\sqrt{r'_t}}\left|\sqrt{\frac{r'_t}{r_t}} - 1\right| \le t^{-2/3}$ and $\sqrt{r_t} \ge t^{4/9}$. Therefore,

$$\begin{aligned}
\left|\frac{1}{\sqrt{r'_t}}\sum_{i=1}^{r'_t-1} Z_i - \frac{1}{\sqrt{r_t}}\sum_{i=1}^{r_t-1} Z_i\right| &\le \left|\frac{1}{\sqrt{r_t}} - \frac{1}{\sqrt{r'_t}}\right| \times \left|\sum_{i=1}^{r'_t-1} Z_i\right| + \frac{1}{\sqrt{r_t}}\left|\sum_{i=1}^{r_t-1} Z_i - \sum_{i=1}^{r'_t-1} Z_i\right|\\
&\le t^{-2/3}t^{3/5} + t^{-4/9}t^{2/5}\\
&= o(1)
\end{aligned}$$

w.h.p. so that

$$\mathbb{P}\left(\frac{\sum_{i=1}^{r_t-1} W_i^{(\epsilon)} - \frac{h_w t}{\tau}}{\sqrt{t}} \leq \lambda\right) = \mathbb{P}\left(\frac{1}{\sqrt{r'_t}} \sum_{i=1}^{r'_t-1} Z_i \leq \frac{\sqrt{\tau}\lambda}{h_w} + R_t^{(3)}\right)$$

where $R_t^{(3)} = o(1)$ with high probability. Applying the CLT to the series associated to the sequence $(Z_i)_{i\geq 1}$ concludes the proof, and we have

$$h_{\mathcal{T}_\mathcal{G}} = \frac{h_w}{\tau} \text{ and } \sigma_{\mathcal{T}_\mathcal{G}}^2 = \frac{h_w^2 \sigma_Z^2}{\tau}.$$

Note that we could not have applied directly the CLT to $\frac{1}{\sqrt{r_t}} \sum_{i=1}^{r_t-1} Z_i$ since r_t is random and not a priori independent of the Z_i's (both depend on the τ_i's). □

Proposition 4.1.20. (Theorems D and E in [117]) *There exist $s, \sigma_s > 0$ such that*

$$\frac{he(\mathcal{X}_t)}{t} \overset{a.s.}{\to} s \tag{4.17}$$

and

$$\frac{he(\mathcal{X}_t) - st}{\sqrt{t}} \overset{law}{\to} \mathcal{N}(0, \sigma_s^2). \tag{4.18}$$

Proof. The proof is similar to that of Theorem 4.1.3. Again, the fact that $\sigma_s > 0$ is due to **A.3**. □

Remark 4.1.21. *The convergences in Theorem 4.1.3 and Proposition 4.1.20 do not depend on the choice of v_*. Moreover, Theorem 4.1.3, Proposition 4.1.20 and Corollary 4.1.16 give*

$$h_{\mathcal{T}_\mathcal{G}} = s h_W. \tag{4.19}$$

We discuss ways of computing $h_{\mathcal{T}_\mathcal{G}}$ and s in the Appendix 1 (Section 4.1.6).

4.1.4 Proofs of Proposition 4.1.2 and Theorem 4.1.1

The lower bound: Proof of Proposition 4.1.2

For all $n \geq 1$, let $\mathcal{G}_n$ be an arbitrary n-lift of $\mathcal{G}$. The proof goes as follows: we couple a RW $(X_t)_{t\geq 0}$ on $\mathcal{G}_n$ with a RW $(\mathcal{X}_t)_{t\geq 0}$ on $\mathcal{T}_\mathcal{G}$. The estimate provided by Theorem 4.1.3 on W_t implies that $\mathcal{X}_t$ is concentrated on $o(n)$ vertices with positive probability for t close to $h^{-1}\log n$, and so is X_t: cycles in $\mathcal{G}_n$ can only reinforce this concentration. This implies a lower bound on $d_x(t)$, where

$$d_x(t) := d_{TV}(P_n^t(x, \cdot), \pi_n), \tag{4.20}$$

since π_n is almost uniform on V_n.

Fix $x \in V_n$ and $\circ \in V_{\mathcal{T}_\mathcal{G}}$ such that the label of $\circ$ is the type of x. Let $(\mathcal{X}_t)_{t\geq 0}$ be a RW on

$(\mathcal{T}_\mathcal{G},\circ)$ starting at the root. We couple $(\mathcal{X}_t)$ with a RW $(X_t)_{t\geq 0}$ on $\mathcal{G}_n$ in the following manner: let $X_0=x$ a.s. For all $t\geq 0$, X_{t+1} is the unique vertex of V_n such that there is an oriented edge from X_t to X_{t+1} whose type is the label of $(\mathcal{X}_t,\mathcal{X}_{t+1})$. Clearly, (X_t) is well defined and is indeed a RW on $\mathcal{G}_n$.
We define a map ϕ from the set of oriented paths starting at x in $\mathcal{G}_n$ to that of oriented paths starting at $\circ$ in $\mathcal{T}_\mathcal{G}$: for all $m\geq 1$ and $p:=(\overrightarrow{e}_1,\dots,\overrightarrow{e}_m)$ an oriented path of length m in $\mathcal{G}_n$, $\phi(p)=(\overrightarrow{e}'_1,\dots,\overrightarrow{e}'_m)$ where $\overrightarrow{e}'_1$ is the unique oriented edge such that its initial vertex is $\circ$, and the label of $\overrightarrow{e}'_1$ is the type of $\overrightarrow{e}_1$, and for all $i\geq 2$, $\overrightarrow{e}'_i$ is the unique edge such that its initial vertex is the end vertex of $\overrightarrow{e}'_{i-1}$ and the label of $\overrightarrow{e}'_i$ is the type of $\overrightarrow{e}_i$.

Remark 4.1.22. *For all $y\in V_{\mathcal{T}_\mathcal{G}}$, if p_1 (resp. p_2) is an oriented path of length $t\geq 1$ from $\circ$ to y, then $\phi^{-1}(p_1)$ and $\phi^{-1}(p_2)$ end at the same vertex of V_n. The converse is not true as soon as $\mathcal{G}_n$ has cycles: for every $x'\in V_n$, there are two distinct non-backtracking paths p_1 and p_2 from x to y', and the oriented paths $\phi(p_1)$ and $\phi(p_2)$ lead to two different vertices y'_1 and y'_2 of $(\mathcal{T}_\mathcal{G},\circ)$.*

Now, let $\lambda>\lambda'\in\mathbb{R}$ and define $t_n:=\lfloor\frac{\log n}{h_{\mathcal{T}_\mathcal{G}}}+\lambda'\frac{\sigma_{\mathcal{T}_\mathcal{G}}}{h_{\mathcal{T}_\mathcal{G}}^{3/2}}\sqrt{\log n}\rfloor$. By Theorem 4.1.3 and Proposition 4.1.20,

$$\liminf_{n\to+\infty}\mathbb{P}(\{W_{t_n}\leq h_{\mathcal{T}_\mathcal{G}}t_n-\lambda\sigma_{\mathcal{T}_\mathcal{G}}\sqrt{t_n}\}\cap\{|he(\mathcal{X}_{t_n})-st_n|>t_n^{2/3})\})\geq\Phi(\lambda). \tag{4.21}$$

For $n\geq 1$, define the sets $A_n:=\{W_{t_n}\leq h_{\mathcal{T}_\mathcal{G}}t_n-\lambda\sigma_{\mathcal{T}_\mathcal{G}}\sqrt{t_n}\}\cap\{|he(\mathcal{X}_{t_n})-st_n|>t_n^{2/3}\}$ and $U_n:=\{y\in V_{\mathcal{T}_\mathcal{G}}\,|\,A_n\cap\{\mathcal{X}_{t_n}=y\}\neq\emptyset\}$. Note that for all $R>0$,

$$\sum_{y\in V_{\mathcal{T}_\mathcal{G}},he(y)=R}\exp(-W(y))=1.$$

Hence for n large enough,

$$\begin{aligned}|U_n|\leq&(2t_n^{2/3}+1)\exp(h_{\mathcal{T}_\mathcal{G}}t_n-\lambda\sigma_{\mathcal{T}_\mathcal{G}}\sqrt{t_n})\\ \leq&\frac{\log n}{h_{\mathcal{T}_\mathcal{G}}}\exp\left(\log n+(\lambda'-\lambda)\frac{\sigma_{\mathcal{T}_\mathcal{G}}}{2\sqrt{h_{\mathcal{T}_\mathcal{G}}}}\sqrt{\log n}\right)\\ \leq&n\exp\left((\lambda'-\lambda)\frac{\sigma_{\mathcal{T}_\mathcal{G}}}{4\sqrt{h_{\mathcal{T}_\mathcal{G}}}}\sqrt{\log n}\right).\end{aligned}$$

Let $B_n=\{x'\in V_n\,|\,A_n\cap\{X_{t_n}=x'\}\neq\emptyset\}$. By Remark 4.1.22, $|B_n|\leq|U_n|$, hence

$$\begin{aligned}d_x(t_n)\geq&\sum_{x'\in B_n}(P_n^{t_n}(x,x')-\pi_n(x'))\\ \geq&\mathbb{P}(A_n)-\pi_{max}n^{-1}|B_n|\\ \geq&\mathbb{P}(A_n)-\pi_{max}\exp\left((\lambda'-\lambda)\frac{\sigma_{\mathcal{T}_\mathcal{G}}}{4\sqrt{h_{\mathcal{T}_\mathcal{G}}}}\sqrt{\log n}\right),\end{aligned}$$

where we recall that $\pi_{max} := \max_{v \in V_{\mathcal{G}}} \pi(v)$. Thus, by (4.21),

$$\liminf_{n \to +\infty} d_x(t_n) \geq \Phi(\lambda).$$

Note that this result is uniform in x, due to Remark 4.1.21. This concludes the proof with $h = h_{\mathcal{T}_{\mathcal{G}}}$ and $\sigma = \frac{\sigma_{\mathcal{T}_{\mathcal{G}}}}{h_{\mathcal{T}_{\mathcal{G}}}^{3/2}}$.

The upper bound: Proof of Theorem 4.1.1

From now on, we focus on the case where $\mathcal{G}_n$ is a uniform random lift of $\mathcal{G}$. The proof consists of three parts, as detailed in Section 4.1.1.

a) Almost mixing for a typical starting point

In this paragraph, we prove that with a large probability, $\mathcal{G}_n$ is such that a RW $(X_t)_{t\geq 0}$ started at a uniformly chosen vertex $x \in V_n$ has a large probability to stay on a subtree T of $\mathcal{G}_n$ for $t'_n := h^{-1} \log n + a\sqrt{\log n}$ steps (hence not seeing cycles), where a is a constant given by Proposition 4.1.23. This allows us to couple $(X_t)_{t\geq 0}$ with a RW $(\mathcal{X}_t)_{t\geq 0}$ on $\mathcal{T}_{\mathcal{G}}$ (keep all the notations of Section 4.1.3 for $(\mathcal{X}_t)$).
As underlined in the Introduction, by Proposition 4.1.4, $(\mathcal{X}_t)$ stays close to its ray to infinity (at distance $O(\log \log n)$ during the first t'_n steps), and this reduces the number of vertices of $\mathcal{G}_n$ that (X_t) could explore. This ray localization also ensures that $W(\mathcal{X}_t)$ and $P_n^t(x, X_t)$ are close quantities, so that for some constant K, for most $x' \in V_n$, $P_n^{t_n}(x, x') \leq \frac{e^{K\sqrt{\log n}}}{n}$, and this allows to deduce Corollary 4.1.5.
Let $\beta > 0$ be an arbitrary positive constant.

The exploration.
Let $n \in \mathbb{N}$ and $\mathcal{N}(\beta) := \{y \in V_{\mathcal{T}_{\mathcal{G}}}, W(y) \geq n^{-1} \exp(\beta\sqrt{\log n})\}$. Set

$$R := \lfloor C \log \log n \rfloor \tag{4.22}$$

with C large enough so that $C_1 \exp(-C_2 R) = o(\log^{-1} n)$. Let $(\mathcal{X}_t)_{t\geq 0}$ be a RW on $(\mathcal{T}_{\mathcal{G}}, \circ)$ started at the root, independent of the realization of $\mathcal{G}_n$. For every $j \geq 1$, let t_j be the hitting time of $B(\circ, R + j)$ by $(\mathcal{X}_t)$. We reveal the structure of $\mathcal{G}_n$ edge by edge, starting from x and making systematic use of Lemma 4.1.6. At every moment of the exploration, let ϕ be an isomorphism between T and a subtree $\mathfrak{T}$ of $(\mathcal{T}_{\mathcal{G}}, \circ)$, and denote L_k and $\mathfrak{L}_k$ the respective height-k levels of T and $\mathfrak{T}$ for all $k \geq 0$.
At step 0, T and $\mathfrak{T}$ are reduced to their respective roots x and $\circ$, so that $\phi(x) = \circ$. Let $X_0 = \phi^{-1}(\mathcal{X}_0) = x$.
Reveal $B(x, R + 1)$. If there is a cycle in $B(x, R + 1)$, stop the exploration. Else, include $B(x, R + 1)$ in T. If $\phi(B(x, R + 1)) \not\subseteq \mathcal{N}(\beta)$, stop the exploration. Else, let $X_t := \phi^{-1}(\mathcal{X}_t)$ for

$t \leq t_1$.

For $j \geq 1$, if the exploration has not been stopped before time t_j, set $x_j := X_{t_j}$ and let α_j be the R-ancestor of x_j. Reveal the pairings of the yet unmatched half-edges of the non-marked vertices of L_{R+j}. Erase the edges closing a cycle, and place a mark at their endpoints. Extend ϕ to L_{R+j+1}, and place a mark at vertices y such that $\phi(y) \notin \mathcal{N}(\beta)$. Let then O_j be the $(R+1)$-offspring of α_j in T. Stop the exploration at time t_j if there is a marked vertex in O_j. Else, for $t_j < t \leq t_{j+1}$, let $X_t := \phi^{-1}(\mathcal{X}_t)$ and stop the exploration at time t if $X_t = \alpha_j$.

Stop the exploration at time t'_n if it was still running.

For $k \geq 1$, we say that the exploration is k-**successful** if it does not stop before time k. If the exploration is k-successful, then $(X_t)_{0 \leq t \leq k}$ is a RW on $\mathcal{G}_n$.

Remark that by construction, T is indeed a tree. Informally, we take at each level all possible vertices v such that $W(\phi(v))$ is above a certain threshold, without creating a cycle. In fact, it will turn out in the sequel that by Theorem 4.1.3, (X_t) has a negligible probability to visit before time t'_n a vertex whose counterpart in $\mathcal{T}_{\mathcal{G}}$ has a small weight (provided a is small enough), and keeping these vertices in T could create cycles, which we want to avoid. This construction of T depends on β (but it will turn out that the choice of β is not important for the results of this Section).

Remark also that the purpose of stopping the exploration if (X_t) visits α_j after x_j is to ensure that $(\mathcal{X}_t)$ stays at distance $\leq R$ of its ray to infinity.

We insist that there are two sources of randomness: the matchings of the half-edges in $\mathcal{G}_n$ and the trajectory of (X_t). Denote $\mathbb{P}_{ann}$ the annealed probability on $\mathcal{G}_n$ and (X_t), and $\mathbb{P}_{\mathcal{G}_n}$ the quenched probability on (X_t) conditionally on the realization of $\mathcal{G}_n$.

Proposition 4.1.23. *Fix $\varepsilon > 0$. There exists $a < 0$ such that for n large enough,*

$$\mathbb{P}_{ann}\left(\textit{the exploration is } t'_n\textit{-successful}\right) \geq 1 - \varepsilon.$$

Proof. Remark that for every $j \geq 1$, if the exploration is t_j-successful, then it is not t_{j+1}-successful only if:

- (X_t) visits α_j between its respective first visits at x_j and x_{j+1}, or
- a cycle appears while matching the $(R+1)$-offspring of α_{j+1}, or
- $W(\phi(\alpha_j)) \leq n^{-1} \exp(\beta\sqrt{\log n}) w_{min}^{-(R+1)}$, or
- $t_{j+1} > t'_n$.

Note in particular that if the third point does not hold, $\phi(O_j) \subseteq \mathcal{N}(\beta)$ and no vertex in O_j is marked because its counterpart would have a too small weight. For all $j \geq 1$, let

$$E_j := \{\text{the exploration stops at a time } t \in \{t_{j-1}+1, \dots, t_j\}\} \cap \{t_j \leq t'_n\}.$$

It is enough to prove that for $a < 0$ small enough, for n large enough, there exists $J_0 > 1$ such that

$$\mathbb{P}_{ann}(\{t_{J_0} \leq t'_n\} \cup \{\cup_{j=1}^{J_0} E_j\}) \leq \varepsilon.$$

For $n \geq 1$, let $J_0 := \lfloor h^{-1} s \log n + \gamma\sqrt{\log n} \rfloor$ where the constant $\gamma \leq 0$ is such that we have $\mathbb{P}(W(\xi_{J_0}) > n^{-1}\exp(\beta\sqrt{\log n})) \geq 1 - \varepsilon/8$. By Corollary 4.1.16 and by (4.19), this is possible if one takes γ small enough. Then if $a < 0$ is small enough, by Proposition 4.1.20, we have that for n large enough,

$$\mathbb{P}_{ann}\left(t_{J_0} \leq t'_n\right) \leq \varepsilon/4.$$

Thus, it remains to prove that

$$\mathbb{P}_{ann}(\cup_{j=1}^{J_0} E_j) \leq 3\varepsilon/4. \tag{4.23}$$

The probability that a cycle arises while revealing $B(x, R+3)$ is at most

$$\Delta^{R+4} \frac{\Delta^{R+4}}{|\overrightarrow{E}_{\mathcal{G}}|n - 2\Delta^{R+4}} = O\left((\log n)^{2C\log\Delta}/n\right).$$

Indeed, the set L_k contains at most Δ^k vertices for all $k \geq 0$, so that we proceed to at most $\Delta + \Delta^2 + \ldots + \Delta^{R+3} \leq \Delta^{R+4}$ pairings of half-edges. Hence for each pairing, there remain at least $|\overrightarrow{E}_{\mathcal{G}}|n - 2\Delta^{R+4}$ unmatched half-edges belonging to vertices not in T, and at most Δ^{R+4} unmatched half-edges belonging to vertices in T.
If $B(x, R+3)$ has no cycle, for n large enough, for every $y \in \phi(B(x, R+2))$,

$$W(y) \geq w_{min}^R > n^{-1}\exp(\beta\sqrt{\log n}),$$

so that there is no marked vertex in $B(x, R+2)$. Thus,

$$\mathbb{P}_{ann}(E_1) = o(1).$$

Now, for $j \geq 1$, let $E'_j \subseteq E_j^c$ be the event that (X_t) visits α_j between its respective first visits at x_j and x_{j+1}, let $E''_j \subseteq E_j^c \cap (E'_j)^c$ be the event that a cycle arises while revealing O_{j+1}, and let $E_j^{(3)} \subseteq E_j^c \cap (E'_j)^c \cap (E''_j)^c$ be the event that $W(\phi(\alpha_j)) \leq n^{-1}\exp(\beta\sqrt{\log n})w_{min}^{-R}$. Remark that we have $E_{j+1} \subseteq E'_j \cup E''_j \cup E_j^{(3)}$.
On E'_j, $(\mathcal{X}_t)_{t \geq t_j}$ visits $\phi(\alpha_j)$, a vertex at distance R of its initial position. Thus, by the second part of Proposition 4.1.13 and our choice for R in (4.22), $\mathbb{P}_{ann}(E'_j) \leq C_1\exp(-C_2 R) = o\left((\log n)^{-1}\right)$, and this bound is uniform in j.
The R-offspring of α_{j+1} contains at most Δ^R vertices. Hence, at most Δ^{R+1} pairings of half-edges are performed to reveal O_{j+1}. We claim that L_{R+j} contains at most $\Delta n\exp(-\beta\sqrt{\log n})$ vertices. If $y \in L_{R+j}$ and y' is its parent, then $\phi(y') \in \mathcal{N}(\beta) \cap \mathfrak{L}_{R+j-1}$. Since

$$\sum_{y'' \in \mathcal{N}(\beta) \cap \mathfrak{L}_{R+j-1}} W(y'') \leq \sum_{y'' \in \partial B(\circ, R+j-1)} W(y'') = 1,$$

and since $W(\phi(y')) \geq n^{-1}\exp(\beta\sqrt{\log n})$, there are at most $n\exp(-\beta\sqrt{\log n})$ such vertices y' (recall that ϕ is a bijection). Each has at most Δ children, and this proves the claim. Thus, the probability to create a cycle while revealing O_{j+1}, i.e. to connect a vertex of O_{j+1} with a vertex of $O_{j+1} \cup L_{R+j}$ is

$$O(n^{-1}\Delta^{R+1}(|L_{R+j}| + |O_{j+1}|)) = O(\Delta^R \exp(-\beta\sqrt{\log n})) = o\left((\log n)^{-1}\right).$$

Hence $\mathbb{P}_{ann}(E''_j) = o\left((\log n)^{-1}\right)$. Again, this is uniform in j.
We have $E^{(3)}_j \subseteq E^*$, with $E^* := \{\exists k \leq t'_n, W(\mathcal{X}_k) \leq n^{-1}\exp(\beta\sqrt{\log n}) w_{min}^{-R}\}$. Note that

$$E^* \subseteq \{\exists k \leq t'_n, d(\mathcal{X}_k, \xi) \geq R\} \cup \{W(\xi_{J_0}) \leq n^{-1}\exp(\beta\sqrt{\log n})\} \cup \{t_{J_0} < t'_n\}.$$

By Proposition 4.1.13 and (4.22), a union bound on all $0 \leq k \leq t'_n$ yields

$$\mathbb{P}_{ann}(\exists k \leq t'_n, d(\mathcal{X}_k, \xi) \geq R) = o(1).$$

Moreover, by our choice of γ and a, we know that

$$\mathbb{P}_{ann}(\{W(\xi_{J_0}) \leq n^{-1}\exp(\beta\sqrt{\log n})\} \cup \{t_{J_0} < t'_n\}) \leq 3\varepsilon/8.$$

Hence, for n large enough, $\mathbb{P}_{ann}(E^*) \leq \varepsilon/2$.
All in all, we have

$$\begin{aligned}\mathbb{P}_{ann}(\cup_{j=1}^{J_0} E_j) &\leq \mathbb{P}_{ann}(E_1) + \sum_{j=1}^{J_0-1} \mathbb{P}_{ann}(E'_j \cup E''_j \cup E^{(3)}_j) \\ &\leq \mathbb{P}_{ann}(E_1) + \sum_{j=1}^{J_0-1} (\mathbb{P}_{ann}(E'_j) + \mathbb{P}_{ann}(E''_j)) \; + \mathbb{P}_{ann}(E^*) \\ &\leq \varepsilon/2 + o(1) \\ &\leq 3\varepsilon/4\end{aligned}$$

for n large enough. This establishes (4.23) and concludes the proof. □

Remark 4.1.24. *For large enough n, less than $n\exp(-2\beta\sqrt{\log n}/3)$ vertices and edges of $\mathcal{G}_n$ are discovered during the exploration. Indeed, as shown in the proof of Proposition 4.1.23, each level of T has at most $\Delta n\exp(-\beta\sqrt{\log n})$ vertices, and there are no more than $h^{-1}\log n$ levels, since we stop the exploration after $t'_n \leq h^{-1}\log n$ steps. Hence, T has less than $n\exp(-3\beta\sqrt{\log n}/5)$ vertices. Each vertex seen in the exploration is either in T or a neighbour of a vertex of T, and every vertex of $\mathcal{G}_n$ has at most Δ edges and Δ neighbours.*

The result of Proposition 4.1.23 is annealed, and leads to a quenched result for "most" realizations of $\mathcal{G}_n$, Corollary 4.1.5.

Proof of Corollary 4.1.5. Let $\varepsilon, K > 0$. Take a such that Proposition 4.1.23 holds. We have $1 - \nu_n(V_n) = \sum_{x' \in V_n} P_n^{t'_n}(x, x') - \nu_n(x')$. Remark that for all $\mathrm{a}, \mathrm{b}, \mathrm{m} \in \mathbb{R}$, such that $\mathrm{a} \geq \mathrm{b}$, $\mathrm{a} - \mathrm{a} \wedge \mathrm{m} \leq (\mathrm{a} - \mathrm{b}) + (\mathrm{b} - \mathrm{b} \wedge \mathrm{m})$, so that

$$
\begin{aligned}
1 - \nu_n(V_n) \leq & \sum_{x' \in V_n \setminus T} P_n^{t'_n}(x, x') - \nu_n(x') \\
& + \sum_{x' \in T} \Big(P_n^{t'_n}(x, x') - P_n^{t'_n}(x, x', T) \Big) + \Big(P_n^{t'_n}(x, x', T) - \nu'_n(x') \Big)
\end{aligned}
$$

where $P_n^k(x, x', T)$ is the probability that a RW on $\mathcal{G}_n$ started at x reaches x' in k steps without leaving the exploration tree T, and $\nu'_n(x') := P_n^{t'_n}(x, x', T) \wedge \frac{\exp(K\sqrt{\log n})}{n}$. Note that if the exploration is t'_n-successful, then $(X_t)_{1 \leq t \leq t'_n}$ stays in T. Applying Markov's inequality to Proposition 4.1.23, we get that with probability at least $1 - \sqrt{\varepsilon}$, $\mathcal{G}_n$ is such that $\mathbb{P}_{\mathcal{G}_n}\big((X_t)_{1 \leq t \leq t'_n} \text{ leaves } T\big) \leq \sqrt{\varepsilon}$. For such $\mathcal{G}_n$,

$$
\sum_{x' \in V_n \setminus T} P_n^{t'_n}(x, x') - \nu_n(x') + \sum_{x' \in T} \Big(P_n^{t'_n}(x, x') - P_n^{t'_n}(x, x', T) \Big) \leq \sqrt{\varepsilon},
$$

thus $1 - \nu_n(V_n) \leq \sqrt{\varepsilon} + S_n$ with $S_n := \sum_{x' \in T} \Big(P_n^{t'_n}(x, x', T) - \nu'_n(x') \Big)$. We now prove that for n large enough and for any realization of $\mathcal{G}_n$ that satisfies $\mathbb{P}_{\mathcal{G}_n}\big((X_t)_{1 \leq t \leq t'_n} \text{ leaves } T\big) \leq \sqrt{\varepsilon}$,

$$
S_n \leq \sqrt{\varepsilon}. \tag{4.24}
$$

Note that for all $x' \in T$, we have $P_n^{t'_n}(x, x', T) = P_{\mathcal{T}_\mathcal{G}}^{t'_n}(\circ, \phi(x'), \mathfrak{T})$ where $P_{\mathcal{T}_\mathcal{G}}^{k}(\circ, \phi(x'), \mathfrak{T})$ is the probability that a RW on $(\mathcal{T}_\mathcal{G}, \circ)$ started at $\circ$ reaches $\phi(x')$ in k steps without leaving $\mathfrak{T}$, the subtree of $(\mathcal{T}_\mathcal{G}, \circ)$ corresponding to T. Thus,

$$
\begin{aligned}
S_n & \leq \sum_{y' \in \mathcal{T}_\mathcal{G}} P_{\mathcal{T}_\mathcal{G}}^{t'_n}(\circ, y', \mathfrak{T}) - \left(P_{\mathcal{T}_\mathcal{G}}^{t'_n}(\circ, y', \mathfrak{T}) \wedge \frac{\exp(K\sqrt{\log n})}{n} \right) \\
& \leq \sum_{y' \in \mathcal{T}_\mathcal{G}} P_{\mathcal{T}_\mathcal{G}}^{t'_n}(\circ, y') - \left(P_{\mathcal{T}_\mathcal{G}}^{t'_n}(\circ, y') \wedge \frac{\exp(K\sqrt{\log n})}{n} \right) \\
& \leq 1 - \sum_{y' \in \mathcal{T}_\mathcal{G}} \left(P_{\mathcal{T}_\mathcal{G}}^{t'_n}(\circ, y') \wedge \frac{\exp(K\sqrt{\log n})}{n} \right) \\
& \leq 1 - \Big(1 - \mathbb{P}_{\mathcal{G}_n}\Big(P_{\mathcal{T}_\mathcal{G}}^{t'_n}(\mathcal{X}_0, \mathcal{X}_{t'_n}) \geq \exp(K\sqrt{\log n})/n \Big) \Big) \\
& \leq \mathbb{P}_{\mathcal{G}_n}\Big(P_{\mathcal{T}_\mathcal{G}}^{t'_n}(\mathcal{X}_0, \mathcal{X}_{t'_n}) \geq \exp(K\sqrt{\log n})/n \Big).
\end{aligned}
$$

Since $(\mathcal{X}_t)_{t \geq 0}$ is independent of the realization of $\mathcal{G}_n$, we have that

$$
\begin{aligned}
\mathbb{P}_{\mathcal{G}_n}\left(P_{\mathcal{T}_\mathcal{G}}^{t'_n}(\mathcal{X}_0, \mathcal{X}_{t'_n}) \geq \frac{\exp(K\sqrt{\log n})}{n} \right) & = \mathbb{P}_{ann}\left(P_{\mathcal{T}_\mathcal{G}}^{t'_n}(\mathcal{X}_0, \mathcal{X}_{t'_n}) \geq \frac{\exp(K\sqrt{\log n})}{n} \right) \\
& = \mathbb{P}\left(P_{\mathcal{T}_\mathcal{G}}^{t'_n}(\mathcal{X}_0, \mathcal{X}_{t'_n}) \geq \frac{\exp(K\sqrt{\log n})}{n} \right)
\end{aligned}
$$

where $\mathbb{P}$ is the probability associated to $(\mathcal{X}_t)$. Let A_n be the R-ancestor of $\mathcal{X}_{t'_n}$. Clearly, $W(A_n) \leq w_{min}^{-R} W(\mathcal{X}_{t'_n}) \leq \exp(\sqrt{\log n}) W(\mathcal{X}_{t'_n})$ for n large enough. Hence if K is large enough, then by Theorem 4.1.3, for large enough n:

$$\mathbb{P}\left(W(A_n) \leq \exp(K\sqrt{\log n}/2)/n\right) \geq 1 - \sqrt{\varepsilon}.$$

And for every $y, y' \in \mathfrak{T}_{\mathcal{G}}$ such that y' is in the R-offspring of y, we have that

$$\{A_n = y,\ A_n \in \xi\} \supseteq \{\mathcal{X}_{t'_n} = y'\} \cap \{(\mathcal{X}_t)_{t \geq t'_n} \text{ does not visit } y\}.$$

Since $d(y, y') = R$, by Proposition 4.1.13 and the Markov property applied to $(\mathcal{X}_t)$ at time t'_n,

$$\mathbb{P}(A_n = y,\ A_n \in \xi) \geq P^{t'_n}_{\mathcal{T}_{\mathcal{G}}}(\mathcal{X}_0, y') \times (1 - C_1 \exp(-C_2 R)),$$

so that $W(A_n) \geq (1 - C_1 \exp(-C_2 R))\, P^{t'_n}_{\mathcal{T}_{\mathcal{G}}}(\mathcal{X}_0, \mathcal{X}_{t_n})$ $\mathbb{P}$-a.s.
Hence, with $\mathbb{P}$-probability at least $1 - \sqrt{\varepsilon}$, $\mathcal{X}_{t'_n}$ is such that

$$P^{t'_n}_{\mathcal{T}_{\mathcal{G}}}(\mathcal{X}_0, \mathcal{X}_{t'_n}) \leq \frac{\exp(K\sqrt{\log n}/2)/n}{1 - C_1 \exp(-C_2 R)} \leq \frac{\exp(K\sqrt{\log n})}{n}.$$

Therefore, $\mathbb{P}_{\mathcal{G}_n}\left(P^{t'_n}_{\mathcal{T}_{\mathcal{G}}}(\mathcal{X}_0, \mathcal{X}_{t'_n}) \geq \exp(K\sqrt{\log n})/n\right) \leq \sqrt{\varepsilon}$ and (4.24) is established. Thus, we have proved that with probability at least $1 - \sqrt{\varepsilon}$, $\mathcal{G}_n$ is such that $\nu_n(V_n) \geq 1 - 2\sqrt{\varepsilon}$. Since $\varepsilon > 0$ was chosen arbitrarily, this concludes the proof. □

b) The last jump for mixing

In this section, we complete the mixing initiated in Corollary 4.1.5, proving a weak version of Theorem 4.1.1, for most starting points of the RW on $\mathcal{G}_n$: for all $\varepsilon > 0$, there exists $K(\varepsilon) > 0$ such that if $S \subseteq V_n$ is the set of vertices s satisfying $|t^{(n)}_s(\varepsilon) - h^{-1}\log n| \leq K(\varepsilon)\sqrt{\log n}$, then $\pi_n(S) \to 1$ as $n \to +\infty$.
We use the fact that w.h.p., $\mathcal{G}_n$ is an **expander**:

Proposition 4.1.25 (Expansion). *Let $L(\mathcal{G}_n) := \min_{S \subseteq V_n,\, \pi_n(S) \leq 1/2} \frac{W(S, S^c)}{\pi_n(S)}$ for $n \geq 1$, where for all $A, B \subseteq V_n$, $W(A, B) := \sum_{x \in A, y \in B, \overrightarrow{e}: x \to y} \pi_n(x) w(\overrightarrow{e})$ is the total weight from A to B. There exists $L > 0$ such that w.h.p. as $n \to +\infty$,*

$$L(\mathcal{G}_n) \geq L. \tag{4.25}$$

We postpone the proof to the end of the section. In the literature, $L(\mathcal{G}_n)$ is usually called the **conductance** of $\mathcal{G}_n$. The largest L such that (4.25) holds is the ***Cheeger constant*** of $(\mathcal{G}_n)_{n \geq 0}$. An interesting application of this property is the contraction of L^2 norms:

Proposition 4.1.26. *There exists $\kappa > 0$, only depending $\mathcal{G}$, such that for all $n, t \geq 1$, and all $x \in V_n$,*

$$Var_{\pi_n}(k^{(n)}_{t,x} - 1) \leq Var_{\pi_n}(k^{(n)}_{t-1,x} - 1)(1 - \kappa L(\mathcal{G}_n)^2),$$

where $k^{(n)}_{t,x}(x') := \frac{P^t_n(x, x')}{\pi_n(x')}$ for $x, x' \in V_n$ and $t \geq 0$, and $Var_{\pi_n}(f) := \sum_{x \in V_n} f(x)^2 \pi_n(x)$ for $f : V_n \to \mathbb{R}$.

This is a classical property. Arguments for the proof can be found in Section 2.3 (see in particular (2.11) and (2.12)) and in Theorem 6.8 of [114]. We stress that it is not necessary for P_n to be reversible, as long as it is lazy.

It remains to link the total variation distance and the L^2 distance. By Corollary 4.1.5, for n large enough and for all $p \geq 0$, with probability at least $1-\varepsilon$, $\mathcal{G}_n$ is such that:

$$\begin{aligned} d_x(t'_n+p) = \|P_n^{t'_n+p}(x,\cdot) - \pi_n(\cdot)\|_{TV} &\leq \varepsilon + \|\nu_n P_n^p(,\cdot) - \pi_n(\cdot)\|_{TV} \\ &\leq \varepsilon + D\|\nu_n P_n^p(,\cdot) - \pi_n(\cdot)\|_{L^2(\pi_n)}, \end{aligned}$$

for some positive constant D (where $\nu_n P_n^p(,y) = \sum_{x' \in V_n} \nu_n(x,x') P_n^p(x',y)$ for every vertex y, and recall the definition of $d_y(t'_n+p)$ from (4.20)). The last inequality is due to Cauchy-Schwarz and the fact that $\pi_{n,min} := \inf_{u \in V_n} \hat{\pi}_n(u) = \pi_{min}/n$. By definition of ν_n, $\|\nu_n - \pi_n\|_{L^2(\pi_n)} \leq \exp(2K\sqrt{\log n})$. Hence by Propositions 4.1.25 and 4.1.26, there exists a constant D' such that for $q \geq D'\sqrt{\log n}$,

$$\|\nu_n P_n^q(,\cdot) - \pi_n\|_{L^2(\pi_n)} \leq \varepsilon, \tag{4.26}$$

so that $d_x(t'_n+q) \leq 2\varepsilon$. This concludes the proof of this weak version of Theorem 4.1.1.

Proof of Proposition 4.1.25. The proof is a corollary from that of Theorem 1 in [20], which states that there exists $\delta > 0$ such that w.h.p.,

$$\min_{S \subseteq V_n,\, |S| \leq |V_n|/2} \frac{|E(S,S^c)|}{|S|} \geq \delta,$$

where $E(S,S^c)$ is the set of non-oriented edges with one endpoint in S and one in its complement. Noticing that $|E(S,S^c)| = |E(S^c,S)|$, one might extend this property in the following way: for all $\theta \in (0,1)$, there exists $\delta(\theta) > 0$ such that w.h.p., $\min_{S \subseteq V_n,\, |S| \leq \theta |V_n|} \frac{|E(S,S^c)|}{|S|} \geq \delta(\theta)$.

Let now $S \subseteq V_n$, such that $\pi_n(S) \leq 1/2$. Recall that the invariant distribution of the RW associated to $\mathcal{G}_n$ is $\pi_n(x) = \pi(u)/n$ for all $x \in V_n$ of type u. Then there exists $\theta_0 \in (0,1)$ such that for any $S \subseteq V_n$, if $\pi_n(S) \leq 1/2$, then $|S| \leq \theta_0 |V_n|$. But $\frac{W(S,S^c)}{\pi_n(S)} \geq \frac{w_{min}\pi_{min}}{\pi_{max}} \frac{|E(S,S^c)|}{|S|}$.

This implies (4.25), with $L \geq \delta(\theta_0) w_{min}\pi_{min}/\pi_{max}$. □

c) Extending the starting point

Fix $\varepsilon \in (0,1)$. Let $r := \lfloor C' \log\log n \rfloor$ for some constant $C' > 0$ such that $r > 2R$, where R was defined in (4.22). Say that $x \in V_n$ is a **root** if $B(x,r)$ contains no cycle, and denote $\mathcal{R}_n$ the set of roots. Denote λ_x the hitting measure on $\partial B(x,r)$ of a RW started at x.

This section is organized as follows: first, we prove that with high probability on $\mathcal{G}_n$, the random walk has a high probability to reach a root in $O(\log\log n)$ steps, uniformly in the starting point (Proposition 4.1.28). Second, we prove that with high probability on $\mathcal{G}_n$, for every root x of $\mathcal{G}_n$, a RW starting at x has a probability at least $1-3\varepsilon$ to leave $B(x,r)$ at a vertex y from which the exploration described in Section 4.1.4 a) is t'_n-successful. This relies on the fact that conditionally on $\{x \in \mathcal{R}_n\}$:

- the exploration from any given vertex $y \in \partial B(x,r)$ has a probability at most ε not to be successful (Lemma 4.1.30), so that the mean value of

$$\lambda_x(y \in \partial B(x,r), \text{ the exploration from } y \text{ is successful})$$

  is at least $1-\varepsilon$,

- and the explorations from two vertices $y, y' \in \partial B(x,r)$ whose common ancestor in $B(x,r)$ is at distance at least R (which is the case for most such couples (y,y')) are almost independent, so that for every $x \in \mathcal{R}_n$, $\lambda_x(y \in \partial B(x,r)$, the exploration from y is successful) is concentrated around its mean.

Third, we give a proof of Theorem 4.1.1 (under **A.3** and **A.4**), using the results of Sections 4.1.4 a) and 4.1.4 b).

Lemma 4.1.27. *W.h.p. as $n \to +\infty$, for every $x \in V_n$, $B(x,5r)$ contains at most one cycle.*

Proof. We proceed via a counting argument similar to the beginning of the proof of Proposition 4.1.23, while estimating the probability that the exploration stops at step 0: take $x \in V_n$, proceed to the $O(\Delta^{5r})$ successive matchings of half-edges to generate $B(x,5r)$. Each matching has a probability $O(\Delta^{5r}/n)$ to close a cycle. Hence, the probability that at least two matchings close a cycle is $O\left((\Delta^{5r})^2 \times (\Delta^{5r}/n)^2\right) = o(1/n)$. By a union bound, $\mathbb{P}(\exists x \in V_n,\, x \notin \mathcal{B}_n) = o(1)$. □

Proposition 4.1.28. *Let $c := \frac{3}{2s}$, where s is the constant of Proposition 4.1.20. For all $\delta > 0$, there exists $n_0 \geq 1$ such that for all $n \geq n_0$ and for all realizations of $\mathcal{G}_n$ such that Lemma 4.1.27 holds,*

$$\max_{x \in V_n} P_n^{\lfloor cr \rfloor}(x, V_n \setminus \mathcal{R}_n) \leq \delta. \tag{4.27}$$

As a consequence, $\max_{x \in V_n} P_n^{\lfloor cr \rfloor}(x, V_n \setminus \mathcal{R}_n) \xrightarrow{\mathbb{P}} 0.$

Proof. Fix $\delta > 0$ and a realization of $\mathcal{G}_n$ such that Lemma 4.1.27 holds, for some $n \geq 0$.
Let $x \in V_n$. If $B(x,5r)$ contains no cycle, then it is isomorphic to $(\mathcal{T}_\mathcal{G}, \circ)$ for some $\circ \in V_{\mathcal{T}_\mathcal{G}}$ on its first $5r$ levels. For n large enough, for all $\circ \in V_{\mathcal{T}_\mathcal{G}}$, $\mathbb{P}(4r/3 \leq he(\mathcal{X}_{\lfloor cr \rfloor}) \leq 5r/3) \geq 1-\delta$ if $(\mathcal{X}_t)_{t \geq 0}$ is a RW on $(\mathcal{T}_\mathcal{G}, \circ)$ started at the root. Hence if $(X_t)_{t \geq 0}$ is a RW on $\mathcal{G}_n$ started at x, $\mathbb{P}(X_{\lfloor cr \rfloor} \in \mathcal{R}_n) \geq 1-\delta$.
Else, there is only one cycle in $B(x,5r)$. Hence $B(x,5r)$ is a cycle C with trees T_v attached to its vertices v of degree at least 3. Let $L \in \mathbb{N}$ be such that for all $\circ \in V_{\mathcal{T}_\mathcal{G}}$, for all $y \in (\mathcal{T}_\mathcal{G}, \circ)$ such that $he(y) \geq L$, $\mathbb{P}(\exists t \geq 0,\, \mathcal{X}_t = y) \leq \delta$ where $(\mathcal{X}_t)_{t \geq 0}$ is a RW on $(\mathcal{T}_\mathcal{G}, \circ)$ started at the root (for any δ, such L exists, by Proposition 4.1.13). Let $C(L)$ be the set of vertices at distance at most L of C, and let $t_0 := \inf\{t \geq 0, X_{t_0} \notin C(L)\}$. We claim that for n large enough, $\mathbb{P}(t_0 > \lfloor \log r \rfloor) \leq \delta$.
By the triangle inequality, all trees in $B(x,5r)$ rooted on C that X_t might visit for some

$t \leq \lfloor \log r \rfloor$ have a maximal height at least $5r - \lfloor \log r \rfloor > L$ for n large enough. Hence for every $0 \leq t \leq \lfloor \log r \rfloor$,

- either $X_t \notin C$, and is at distance less than L from $B(x, 5r) \setminus C(L)$,
- or it is on C, and it is at distance at most $|V|$ from a vertex v of C of degree at least 3, so that v is the root of a tree planted on C: a cycle in $\mathcal{G}$ cannot contain a path of $|V+1|$ consecutive vertices of degree less than 3, and this holds for any n-lift of $\mathcal{G}_n$.

Therefore, we have $d(X_t, B(x, 5r) \setminus C(x, L)) \leq |V| + L$. Hence $\mathbb{P}(X_{t+|V|+L} \notin C(x, L)) \geq w_{min}^{|V|+L}$ for all $0 \leq t \leq \log r$. Decomposing $\{1, \lfloor \log r \rfloor\}$ into intervals of length $|V| + L$, we have

$$\mathbb{P}(X_1, X_2, \ldots, X_{\lfloor \log r \rfloor} \in C(L)) \leq \left(1 - w_{min}^{|V|+L}\right)^{\lfloor \log r/(|V|+L) \rfloor} \leq \delta$$

for n large enough, and this proves the claim.
Suppose now that $t_0 \leq \lfloor \log r \rfloor$. Note that there exists some (random) $y \in C$ such that X_{t_0} is on T_y. As remarked in the proof of the claim, T_y has height at least $5r - \lfloor \log r \rfloor$. Moreover, on its first $5r - \lfloor \log r \rfloor$ levels, (T_y, y) is isomorphic to a copy of $(\mathcal{T}_{\mathcal{G}}, \circ_y)$ where $\circ_y$ is the type of y, from which two branches starting at $\circ_y$ have been removed (corresponding to the two edges from y in C). Recall that $d(X_{t_0}, C) \geq L+1$. By definition of L and Proposition 4.1.20, for n large enough, with probability at least $1 - 2\delta$, (X_t) does not visit y (and hence C) for $t_0 \leq t \leq t_0 + cr$, and we have $4r/3 \leq d(X_{t_0+cr}, C) \leq 5r/3$. On this event and on $\{t_0 \leq \lfloor \log r \rfloor\}$, $B(X_{\lfloor cr \rfloor}, r)$ contains no cycle so that $X_{\lfloor cr \rfloor} \in \mathcal{R}_n$. Therefore, on this realization of $\mathcal{G}_n$, $\mathbb{P}_{\mathcal{G}_n}(X_{\lfloor cr \rfloor} \in \mathcal{R}_n) \geq 1 - 3\delta$ uniformly in the starting point x.
Hence, we have shown that for all $\delta > 0$, for n large enough and $\mathcal{G}_n$ such that $\mathcal{B}_n = V_n$,

$$\max_{x \in V_n} P_n^{\lfloor cr \rfloor}(x, V_n \setminus \mathcal{R}_n) \leq 3\delta.$$

(4.27) follows, and we conclude by Lemma 4.1.27. □

For $x \in \mathcal{R}_n$, let $\mathbb{P}_x$ be the probability distribution of $\mathcal{G}_n$ conditionally on the fact that x is a root, and for all $y \in \partial B(x, r)$, denote $\alpha(y, x)$ the vertex at distance R of y on its shortest path to x.
In addition, if at most $n \exp(-\beta\sqrt{\log n}/2)$ edges of $\mathcal{G}_n$ have been revealed (where β is the constant fixed at the beginning of Section 4.1.4), and if all revealed paths starting from y and leading to revealed cycles go through $\alpha(y, x)$, say that $\mathcal{G}_n$ is a **good context for** y.
Define the **cut exploration from** y as the exploration performed in 4.1.4, except that some matchings may have already been revealed, and give a mark to $\alpha(y, x)$ (hence don't explore the offspring of $\phi(\alpha)$ and stop the exploration if the RW hits $\alpha(y, x)$). If the cut exploration from y is t'_n-successful, we say that it is **good**.

Proposition 4.1.29. *With high probability on $\mathcal{G}_n$, for all $x \in \mathcal{R}_n$,*

$$\lambda_x(\{y| \textit{ the cut exploration from } y \textit{ is good}\}) \geq 1 - 3\varepsilon.$$

To prove this statement, notice first that a cut-exploration from a good context has a large probability to be good.

Lemma 4.1.30. *For $n \in \mathbb{N}$, let $x \in \mathcal{R}_n$ and $y \in \partial B(x,r)$. Suppose that $\mathcal{G}_n$ has been partially revealed and is a good context for y. Then we have*
$\mathbb{P}$ *(the cut exploration from y is good)* $\geq 1-\varepsilon$.

Proof. One checks that all the arguments in the proof of Proposition 4.1.23 remain valid. In particular, the probability of creating a cycle while matching the $(R+1)$-offspring of the R-ancestor of the RW in the first t'_n steps is $o(1)$ and on the event that the exploration is not stopped, the RW has a probability $o(1)$ to visit $\alpha(y,x)$. $\square$

Proof of Proposition 4.1.29. Let x be a root. Let $\alpha_1,\dots,\alpha_q$ be the vertices of $\partial B(x,r-R)$, with $q=|\partial B(x,r-R)|$. For all $i \in \{1,q\}$, let A_i be the set of vertices $y \in \partial B(x,r)$ such that α_i is on the shortest path from x to y. We say that A_i is **intact** whenever for all $y \in A_i$, for all $j<i$ and $y' \in A_j$, no vertex in the cut exploration from y' is matched to y. On the contrary, if such a matching exists, say that the relevant edge is a **mismatch**. Denote V_α the set of vertices α_i such that A_i is not intact. Denote V_{mis} (resp. B) the set of $y \in \partial B(x,r)$ which are not intact (resp. which are intact and whose cut exploration is not good). It is enough to prove that

$$\mathbb{P}_x(\lambda_x(B \cup V_{mis}) \geq 3\varepsilon) = o(1/n),$$

where the $o(1/n)$ is uniform in $x \in \mathcal{R}_n$.
Let I_n be the cardinality of V_α, and J_n the number of mismatches. Clearly, $I_n \leq J_n$ a.s. and there exists $K'>0$, independent of x, such that for large enough n, $|\partial B(x,r)| \leq \log^{K'} n$. Hence by Remark 4.1.24, while performing the cut explorations of all $y \in A_i$ for all $i \leq q$, less than $n\exp(-\beta\sqrt{\log n}/2)$ edges are created for n large enough. Edges arise from independent matchings, so that for n large enough, J_n is stochastically dominated by a sum of $n\exp(-\beta\sqrt{\log n}/2)$ independent Bernoulli random variables of parameter $2\log^{K'} n/n$. This entails for all integers $U \geq 1$:

$$\begin{aligned}\mathbb{P}_x(I_n \geq U) \leq \mathbb{P}_x(J_n \geq U) &\leq \binom{n\exp(-\beta\sqrt{\log n}/2)}{U}\left(\frac{2\log^{K'} n}{n}\right)^U \\ &\leq \left(\frac{2\log^{K'} n}{\exp(\beta\sqrt{\log n}/2)}\right)^U\end{aligned}$$

since $\binom{M}{N} \leq M^N$ for $M,N \in \mathbb{N}$. Letting $U=\lfloor 3\sqrt{\log n}/\beta \rfloor$, we obtain $\mathbb{P}_x(I_n \geq U)=o(1/n)$, and this is uniform in $x \in \mathcal{R}_n$. But if $I_n \leq \lfloor 3\sqrt{\log n}/\beta \rfloor$, by Proposition 4.1.13,

$$\begin{aligned}\lambda_x(V_{mis}) &\leq \mathbb{P}((X_t) \text{ hits a non-intact } \alpha_i \text{ before leaving } B(x,r)\ ) \\ &= O\left(\sqrt{\log n}e^{-C_2(r-R)}\right).\end{aligned}$$

For C' large enough in the definition of r, $\sqrt{\log n}e^{-C_2(r-R)} = o(1)$.
Therefore,

$$\mathbb{P}_x(\lambda_x(V_{mis}) \geq \varepsilon) = o(1/n)$$

uniformly in $x \in \mathcal{R}_n$. Hence, it remains to prove that $\mathbb{P}_x(\lambda_x(B) > 2\varepsilon) = o(1/n)$.
For $i \leq q$, let $\mathcal{F}_i$ be the σ-field generated by the cut explorations of the vertices in $\cup_{j=1}^{i} A_j$, and define $\mathcal{Y}_i := \lambda_x(B \cap A_i)$, $\mathcal{Z}_i := \mathcal{Y}_i - \mathbb{E}_x(\mathcal{Y}_i | \mathcal{F}_{i-1})$ and $\mathcal{W}_i := \sum_{j=1}^{i} Z_j$. According to Lemma 4.1.30, for all $i \leq q$,

$$\begin{aligned}
\mathbb{E}_x(\mathcal{Y}_i | \mathcal{F}_{i-1}) &= \mathbf{1}_{\{A_i \text{ is intact}\}} \sum_{y \in A_i} \lambda_x(y) \mathbb{P}_x(B_y | \mathcal{F}_{i-1}) \\
&\leq \varepsilon \lambda_x(A_i),
\end{aligned}$$

where $B_y = \{\text{the cut-exploration from } y \text{ is not good}\}$. In particular, $\lambda_x(B) \leq \varepsilon + W_q$. And for all $i \geq 0$,

$$\begin{aligned}
(\mathcal{W}_{i+1} - \mathcal{W}_i)^2 = \mathcal{Z}_{i+1}^2 &\leq 2(\mathcal{Y}_{i+1}^2 + \mathbb{E}_x(\mathcal{Y}_{i+1} | \mathcal{F}_i)^2) \\
&\leq 2\lambda_x(A_{i+1})^2 + 2\varepsilon^2 \lambda_x(A_{i+1})^2 \\
&\leq 4\lambda_x(A_{i+1})^2,
\end{aligned}$$

so that

$$\sum_{i=0}^{q-1} (\mathcal{W}_{i+1} - \mathcal{W}_i)^2 \leq 4 \sum_{j=1}^{q} \lambda_x(A_j)^2 \leq 4 \max_{1 \leq i \leq q} \lambda_x(A_i).$$

By Proposition 4.1.13, $\max_{1 \leq i \leq q} \lambda_x(A_i) \leq C_1 e^{-C_2(r-R)}$. We apply Azuma-Hoeffding's inequality to the martingale $(\mathcal{W}_i)_{1 \leq i \leq q}$ to get

$$\mathbb{P}_x(\mathcal{W}_q \geq \varepsilon) \leq \exp\left(-\frac{\varepsilon^2}{2C_1 e^{-C_2(r-R)}}\right) = o(1/n),$$

so that $\mathbb{P}_x(\lambda_x(B) > 2\varepsilon) = o(1/n)$, the $o(1/n)$ being uniform in all $x \in \mathcal{R}_n$. This concludes the proof. □

From this and the conclusions of 4.1.4 b), we deduce the following:

Corollary 4.1.31. *For every $\varepsilon > 0$, with high probability on $\mathcal{G}_n$, for all $x \in \mathcal{R}_n$,*

$$\lambda_x\left(\{y \,|\, d_y(\lfloor t'_n + D'\sqrt{\log n}\rfloor) \leq \varepsilon\}\right) \geq 1 - \varepsilon,$$

where D' is defined as in (4.26) and d_y is defined in (4.20).

Proof of Theorem 4.1.1. Let $\varepsilon > 0$. Define $T_n := t'_n + D'\sqrt{\log n} + cr + 2s^{-1}r$. For any realization of $\mathcal{G}_n$ and all vertices $x, y \in V_n$, $P_n^{T_n}(x, y) = \sum_{x' \in V_n} P_n^{cr}(x, x') P_n^{T_n - cr}(x', y)$, so that the distance to equilibrium starting from x satisfies

$$\begin{aligned} 2d_x(T_n) &= \sum_{y\in V_n}\left|\sum_{x'\in V_n} P_n^{cr}(x,x')P_n^{T_n-cr}(x',y)-\pi_n(y)\right| \\ &\leq \sum_{x'\in V_n} P_n^{cr}(x,x')\|P_n^{T_n-cr}(x',\cdot)-\pi_n\|_1 \end{aligned}$$

by the triangle inequality. Thus by Proposition 4.1.28, w.h.p. as $n\to+\infty$, $\mathcal{G}_n$ is such that for all $x\in V_n$,

$$d_x(T_n)\leq \varepsilon+\sup_{x'\in\mathcal{R}_n} d_{x'}(T_n-cr). \tag{4.28}$$

Fix a realization of $\mathcal{G}_n$. For all $x'\in\mathcal{R}_n$, again by the triangle inequality,

$$d_{x'}(T_n-cr)\leq \sum_{y\in\partial B(x',r)}\left(\sum_{\ell=0}^{2s^{-1}r} P_n^{\ell}(x',y)d_z(T_n-cr-\ell)\right)+\mathbb{P}_{\mathcal{G}_n}(\tau_e(x')>2s^{-1}r)$$

where $\tau_e(x')$ is the hitting time of $\partial B_{\mathcal{G}_n}(x',r)$ by a RW started at x'. But $t\mapsto d_y(t)$ is a non-increasing function for all $y\in V_n$, so that

$$d_{x'}(T_n-cr)\leq \mathbb{P}_{\mathcal{G}_n}(\tau_e(x')>2s^{-1}r)+\sum_{y\in\partial B(x',r)}\lambda_{x'}(z)d_y(t'_n+D'\sqrt{\log n}).$$

By Corollary 4.1.31, w.h.p. $\mathcal{G}_n$ is such that $\sum_{y\in\partial B(x',r)}\lambda_{x'}(y)d_y(t'_n+D'\sqrt{\log n})\leq 2\varepsilon$. Moreover, by Proposition 4.1.20, for n large enough, $\mathbb{P}_{\mathcal{G}_n}(\tau_e(x')>2s^{-1}r)\leq\varepsilon$ for all $x'\in\mathcal{R}_n$: since $B_{\mathcal{G}_n}(x',r)$ is isomorphic to the ball of radius r in $\mathfrak{T}$ by definition of $\mathcal{R}_n$, a RW on $B_{\mathcal{G}_n}(x',r)$ behaves like a RW on $\mathfrak{T}$ until it hits $\partial B_{\mathcal{G}_n}(x',r)$. Combining this with (4.28), we obtain that w.h.p., $\mathcal{G}_n$ is such that

$$\sup_{x\in V_n} d_x(T_n)\leq 3\varepsilon.$$

This concludes the proof of Theorem 4.1.1 when **A.3** and **A.4** hold. □

4.1.5 Relaxing the assumptions

We now establish that assumptions **A.3** and **A.4** are not necessary for Proposition 4.1.2 and Theorem 4.1.1.

Getting rid of A.3

Without loss of generality, we can assume that each edge admits at least one orientation with positive weight. Suppose that **A.3** does not hold, hence, that at least one edge has exactly one orientation $\overrightarrow{e}$ with positive weight. In this case, the RW on $(\mathcal{T}_{\mathcal{G}},\circ)$ is not irreducible any more. It is still transient by Proposition 4.1.10. Moreover, assumptions **A.1** and **A.2** are enough to imply that the RW can reach every isomorphism class of subtrees, independently of the choice

of $\circ$, so all constants in Section 4.1.3 can be made independent of v_* and $\overrightarrow{e}_*$.
As a consequence, all results of Section 4.1.3 hold, with the following exceptions. Introduce the assumption (weaker than **A.3**):

A.3* At least one edge has both orientations with positive weight.

Then

- if **A.3*** does not hold, $(\mathcal{X}_t)$ can not backtrack and Proposition 4.1.20 becomes $he(\mathcal{X}_t) = t$ a.s. for all t (hence $s = 1$),
- if **A.3*** does not hold and $\sigma_W = 0$ in Corollary 4.1.16, there exists $K_{\mathcal{T}_\mathcal{G}}$ only depending on $\mathcal{G}$ such that Theorem 4.1.3 becomes $\sup_{t\in\mathbb{N}} |W_t - h_{\mathcal{T}_\mathcal{G}} t| \leq K_{\mathcal{T}_\mathcal{G}}$. Indeed, $\mathcal{X}_t = \xi_t$ for all $t \geq 1$ so that $W_t = W(\xi_t)$ and $h_{\mathcal{T}_\mathcal{G}} = h_W$, and by Lemma 4.1.8, $\sigma_W = 0$ implies that $|W_t - h_{\mathcal{T}_\mathcal{G}} t|$ is bounded.

This implies straightforwardly Proposition 4.1.2, possibly with $\sigma = 0$. One checks readily that the reasoning of Sections 4.1.4 a), 4.1.4 b) and 4.1.4 c) is still true, so that Theorem 4.1.1 holds. For Proposition 4.1.25 in particular, the original proof of Amit and Linial for simple non-directed graphs is based on an argument that only requires **A.1** and **A.2**: pick $V' \subseteq V_n$, that contains say k vertices of a given type u. Those vertices lead to k vertices of type v by irreducibility of the RW associated to $\mathcal{G}$. Hence if V' has less than $k(1-\varepsilon)$ vertices of type v, $W(V', V'^c)/\pi_n(V') \gtrsim \varepsilon$, and we can suppose that every type is represented in V' 'almost' in the same proportion as the others. By **A.2**, there exist two cycles C_1 and $C_2 \neq C_1^{-1}$ of $\mathcal{G}$ such that from each of those k vertices of type u, we can go along two trajectories featuring the types of C_1 and C_2 respectively. If we want $W(V', V'^c)/\pi_n(V') \lesssim \varepsilon$, then at least $(1-\varepsilon)k$ of the k C_1-like (resp. C_2-like) trajectories should end in those k vertices of type u. Denote E_1 (resp. E_2) this event. Remark that we can suppose that C_1 and C_2 don't lie on the same set of non-oriented edges, so that E_1 and E_2 are independent, and the probability that E_1 happens is the probability that a uniform permutation of $\{1, \ldots, n\}$ sends at least $k(1-\varepsilon)$ elements of $\{1, \ldots, k\}$ in $\{1, \ldots, k\}$. An estimation of this quantity via a union bound on the choice of the $k(1-\varepsilon)$ elements and Stirling's formula, and a union bound over all possible subsets $V' \subseteq V_n$ of cardinality at most $V_n/2$ finishes the proof.
The fact that $B(x, R)$ contains more vertices than

$$\widetilde{B}(x, R) := \{y | \text{there is an oriented path of length at most } R \text{ from } x \text{ to } y\}$$

does not change anything to our reasoning (we only need that $|\widetilde{B}(x, R)|$ grows exponentially with R, which is the case by **A.2**).

Getting rid of A.4

As in Lemma 4.1.11, one can decompose $\mathcal{G}$ into a core $c(\mathcal{G})$ satisfying **A.4** and "branches" planted on this core. A similar decomposition holds for $\mathcal{G}_n$, as a n-lift $c(\mathcal{G})_n$ of $c(\mathcal{G})$ and branches isomorphic to those of $c(\mathcal{G})$. By Lemmas 4.1.7 and 4.1.8, after t_0 steps of a RW $(X_t)_{t\geq 0}$ in $\mathcal{G}_n$, the number $n(t_0)$ of steps in $c(\mathcal{G})_n$ follows a CLT: $(n(t_0) - \mathfrak{a}t_0)/\sqrt{t_0} \to \mathcal{N}(0, \mathfrak{b}^2)$ for some constants $\mathfrak{a} > 0, \mathfrak{b} \geq 0$ depending on $\mathcal{G}$ only. Moreover, the trajectory of (X_t) on $c(\mathcal{G})_n$ is a RW associated to $c(\mathcal{G})_n$ by the strong Markov property, so that Theorem 4.1.1 holds (h is replaced by $h\mathfrak{a}$). As for Proposition 4.1.2, the excursion theory presented in Section 4.1.3 is still true, and so is Proposition 4.1.2.

4.1.6 Appendix 1: computing $h = h_{\mathcal{T}_\mathcal{G}}$ and s

In general, computing exact values for $h_{\mathcal{T}_\mathcal{G}}$ and s is a difficult problem. Nagnibeda and Woess [117] give two formulas for s depending on Green functions. The first one (equation (5.8)) is

$$s^{-1} = \sum_{\overrightarrow{e} \in \overrightarrow{E}_\mathcal{G}} \hat{\pi}(\overrightarrow{e}) \frac{F_{\overrightarrow{e}^{-1}}(1)}{w(\overrightarrow{e}^{-1})(1 - F_{\overrightarrow{e}^{-1}}(1))},$$

where $\hat{\pi}$ is the invariant distribution of the NBRW associated to $\widehat{\mathcal{G}}$ (defined in Section 4.1.3), and

$$F_{\overrightarrow{e}}(z) = F_{x,y}(z) := \sum_{n\geq 0} \mathbb{P}(\mathcal{X}_n = y \text{ and } \nexists k < n,\, \mathcal{X}_k = y | \mathcal{X}_0 = x) z^n,$$

for all $z \in \mathbb{C}$ and $x, y \in V_{\mathcal{T}_\mathcal{G}}$ such that the oriented edge (x, y) exists and has label $\overrightarrow{e}$, is the "first passage" Green function. Note that the series $F_{\overrightarrow{e}}$ has a positive radius of convergence. The functions $(F_{\overrightarrow{e}})_{\overrightarrow{e} \in \overrightarrow{E}_\mathcal{G}}$ satisfy a non-linear system of equations given by

$$F_{\overrightarrow{e}^{-1}}(z) = zw(\overrightarrow{e}^{-1}) + z \sum_{\overrightarrow{e}' \leftarrow \overrightarrow{e}} w(\overrightarrow{e}') F_{\overrightarrow{e}'^{-1}}(z) F_{\overrightarrow{e}^{-1}}(z)$$

for all $\overrightarrow{e} \in \overrightarrow{E}_\mathcal{G}$ (Proposition 2.5 in [117]), where $\overrightarrow{e}' \leftarrow \overrightarrow{e}$ means that the end vertex of $\overrightarrow{e}$ is the initial vertex of e'. One can establish this by decomposing the trajectory of a RW $(\mathcal{X}_t)_{t\geq 0}$ as follows: if x, y are such that (x, y) has label $\overrightarrow{e}$, and if $\mathcal{X}_t = y$, then either $\mathcal{X}_{t+1} = x$, or $\mathcal{X}_{t+1} = y'$ for some $y' \in \mathcal{T}_\mathcal{G} \setminus \{x\}$, which happens with probability $w(\overrightarrow{e}')$ where $\overrightarrow{e}'$ is the label of (y, y'). Now, $(\mathcal{X}_t)$ can come back to x in k steps only by reaching y in $k' \leq k - 1$ steps and then reaching x in $k - k'$ steps.

Letting $q(x, y) := F_{(x,y)}(1)$ the probability that $(\mathcal{X}_t)$ reaches y at least once if it starts at x, the transition matrix $\widehat{Q}$ of the NBRW on $\mathcal{T}_{\widehat{\mathcal{G}}}$ satisfies

$$\widehat{Q}(\overrightarrow{e}, \overrightarrow{e}') = \frac{w(\overrightarrow{e}')(1 - q(\overrightarrow{e}'^{-1}))}{\sum_{\overrightarrow{e}'' \leftarrow \overrightarrow{e}, \overrightarrow{e}'' \neq \overrightarrow{e}} w(\overrightarrow{e}'')(1 - q(\overrightarrow{e}''^{-1}))}. \tag{4.29}$$

Indeed, if (x, y) and (y, y') have respective labels $\overrightarrow{e}$ and $\overrightarrow{e}'$, if $(\mathcal{X}_t)$ starts at y, in order to leave to infinity through y':

- either it goes through (y, y') and never returns back from w to v: this has probability $w(\overrightarrow{e}')(1 - q(\overrightarrow{e}'^{-1}))$,

- or it goes through any (y, y''), comes back to y, and will leave to infinity through y': this has probability $\sum_{\overrightarrow{e}'' \leftarrow \overrightarrow{e}} w(\overrightarrow{e}'')q(\overrightarrow{e}''^{-1})\hat{w}(\overrightarrow{e}')$.

Recall that $\hat{w}(\overrightarrow{e}')$ is the probability that $(\mathcal{X}_t)$ leaves to infinity through y. Hence

$$\hat{w}(\overrightarrow{e}') = w(\overrightarrow{e}')(1 - q(\overrightarrow{e}'^{-1})) + \sum_{\overrightarrow{e}'' \leftarrow \overrightarrow{e}} w(\overrightarrow{e}'')q(\overrightarrow{e}''^{-1})\hat{w}(\overrightarrow{e}'),$$

thus

$$\hat{w}(\overrightarrow{e}') = \frac{w(\overrightarrow{e}')(1 - q(\overrightarrow{e}'^{-1}))}{1 - \sum_{\overrightarrow{e}'' \leftarrow \overrightarrow{e}} w(\overrightarrow{e}'')q(\overrightarrow{e}''^{-1})}.$$

Recall that $\widehat{Q}(\overrightarrow{e}, \overrightarrow{e}') = \frac{\hat{w}(\overrightarrow{e}')}{1-\hat{w}(\overrightarrow{e}^{-1})}$, from which we derive the formula (4.29). Note that $\sum_{\overrightarrow{e}'' \leftarrow \overrightarrow{e}} w(\overrightarrow{e}'')q(\overrightarrow{e}''^{-1}) < 1$ since $q(\overrightarrow{g}) < 1$ for all $\overrightarrow{g} \in \overrightarrow{E}_{\mathcal{G}}$. Then, it remains to compute the unique invariant probability measure of $\widehat{Q}$.

Knowing $(F_{\overrightarrow{e}}(1))_{\overrightarrow{e} \in \overrightarrow{E}_{\mathcal{G}}}$ and $\hat{\pi}$, one can compute s. However, those quantities are the solutions of non-linear systems of equations, for which no explicit general solutions have been found. Even for the seemingly simple case where $\mathcal{G}$ has only one oriented cycle (and the inverse cycle), continuous fractions are involved to derive non simple expressions for the $F_{\overrightarrow{e}}$'s in [137]. The second formula for the speed in [117] is derived from a powerful theorem in [127], and involves again those series. For the SRW on periodic trees, Takacs obtains similar equations and gives explicit solutions for three particular cases (examples 4.7, 4.8 and 4.10 in [133]).

h_W has a simple expression in terms of $\widehat{Q}$ and $\hat{\pi}$, namely

$$h_W = \sum_{\overrightarrow{e} \in \overrightarrow{E}_{\mathcal{G}}} \hat{\pi}(\overrightarrow{e})(\log(1 - \hat{w}(\overrightarrow{e})) - \log \hat{w}(\overrightarrow{e}), \tag{4.30}$$

hence once one manages to compute s, computing $h_{\mathcal{T}_{\mathcal{G}}}$ is straightforward (recall that $h_{\mathcal{T}_{\mathcal{G}}} = sh_W$).

Gilch [81] studies transient random walks in the more general context of regular languages, and obtains a law of large numbers:

$$-\log P^t_{\mathcal{T}_{\mathcal{G}}}(\mathcal{X}_0, \mathcal{X}_t)/t \overset{a.s.}{\to} h' \tag{4.31}$$

for some positive h' if the random walk is transient. He proves that h' is an analytic function of the weights in $\mathcal{G}$ and discusses the possibilities to compute the entropy. He obtains a formula that reduces to (4.19) in our particular context: h' is given as the product of three factors, $h' = \lambda^{-1}\ell\mathfrak{h}(Y)$ (Theorem 2.5), where in our particular setting, $\lambda = 1$ is the expected distance between $\mathcal{X}_{\theta_i}$ and $\mathcal{X}_{\theta_{i+1}}$, $\mathfrak{h}(Y) = h_W$ and $\ell = s$ is the rate of escape, whose computation in [80] is the equivalent of that of [117] for regular languages.

Note that in our setting, (4.31) can be deduced from Kingman's subadditive ergodic theorem. We have $P^t_{\mathcal{T}_{\mathcal{G}}}(\mathcal{X}_0, \mathcal{X}_t) \geq P^s_{\mathcal{T}_{\mathcal{G}}}(\mathcal{X}_0, \mathcal{X}_s)P^{t-s}_{\mathcal{T}_{\mathcal{G}}}(\mathcal{X}_s, \mathcal{X}_t)$ a.s. for all $0 \leq s \leq t$. Moreover, if the label of $\mathcal{X}_0$ is distributed according to π (hence, so is the label of $\mathcal{X}_r$ for all $r \in \mathbb{N}$), the random variables $P^{t-s}_{\mathcal{T}_{\mathcal{G}}}(\mathcal{X}_s, \mathcal{X}_t)$ and $P^{t-s}_{\mathcal{T}_{\mathcal{G}}}(\mathcal{X}_0, \mathcal{X}_{t-s})$ have the same distribution that only depends on $t - s$, and we can apply Kingman's theorem to obtain (4.31). Since π is strictly positive on all labels and the convergence is a.s., the result holds if the label of $\mathcal{X}_0$ is chosen arbitrarily. Note that this does not imply that $h' > 0$.

Remark 4.1.32. *Comparing the formula in [81] with (4.19), we get $h' = h_{\mathcal{T}_\mathcal{G}}$. However, one can prove this equality without using the results of [81]. The fact that $h_{\mathcal{T}_\mathcal{G}} \leq h'$ is a consequence from the ray localization of Proposition 4.1.4. Indeed, for all $\varepsilon > 0$ w.h.p. as t goes to infinity, $P^t_{\mathcal{T}_\mathcal{G}}(\mathcal{X}_0, \mathcal{X}_t) \geq \exp(-(h'+\epsilon)t)$, and $(\mathcal{X}_n)_{n\geq t}$ has a probability $o(1)$ to visit the $(\log\log t)$-ancestor of $\mathcal{X}_t$ (the $o(1)$ is uniform conditionally on $\mathcal{X}_t$, by Proposition 4.1.4), so that w.h.p., this ancestor and $\mathcal{X}_t$ itself have a probability at least $\exp(-(h'+2\varepsilon)t)$ to be in the ray to infinity, so that $h_{\mathcal{T}_\mathcal{G}} \leq h' + 2\varepsilon$.*
Conversely, by (4.31), there exists a set S_t of at least $\exp((h'-\varepsilon)t)$ vertices, such that $X_t \in S_t$ w.h.p. as $t \to +\infty$ and such that for all $x \in S_t$, $\exp(-(h'+\varepsilon)t) \leq P^t_{\mathcal{T}_\mathcal{G}}(\mathcal{X}_0, x) \leq \exp(-(h'-\varepsilon)t)$. By (4.18), we can impose that $st - \varepsilon t^{2/3} \leq he(x) \leq st + \varepsilon t^{2/3}$. The fact that the vertices of S_t are localized in less than $3\varepsilon t^{2/3}$ consecutive levels implies that we can decompose $S_t = \sqcup_{i=1}^m S'_m$ for some $m \geq |S_t|/\Delta^{3\varepsilon t^{2/3}} \geq \exp((h'-2\varepsilon)t)$ for t large enough, where S'_i is a subtree of $(\mathcal{T}_\mathcal{G}, \circ)$ with some root x_i, and of height less than $3\varepsilon t^{2/3}$. x_i maximizes $W(x)$ for $x \in S'_i$, which implies that $W(x_i) \geq \exp(-(h_{\mathcal{T}_\mathcal{G}} - \varepsilon)t)$ for all $i \leq m$, and that $\sum_{1\leq i\leq m} W(x_i) \geq m\exp(-(h_{\mathcal{T}_\mathcal{G}} - \varepsilon)t)$. Moreover, we can impose that x_i is in the offspring of no other x_j, so that $\sum_{1\leq i\leq m} W(x_i) \leq 1$. From this, we deduce that $h_{\mathcal{T}_\mathcal{G}} \geq h' - 3\varepsilon$. Further details are left to the reader.

4.1.7 Appendix 2: is laziness necessary?

It is easily checked that the fact that $\alpha > 0$ is not necessary except for the proof of Proposition 4.1.26, so that the rest of our reasoning still holds for $\alpha = 0$ with minor changes (for instance, one might have $\sigma = 0$ in Proposition 4.1.2 if $\mathcal{G}$ does not satisfy **A.3*** and if $\mathcal{T}_\mathcal{G}$ has a cylindrical symmetry, and the RW on $\mathcal{G}$ might have a period $d > 1$, in which case one should look at the RW $(X_t)_{t\geq 0}$ at times $\{t = kd + r,\, k \in \mathbb{N}\}$, for each residue r modulo d, details are left to the reader).
A sufficient condition to guarantee that the results of [114] required for the proof of Proposition 4.1.26 hold would be that there exists $c > 0$ such that for all n large enough,

$$\inf_{f, Var_{\pi_n}(f)=1} \mathcal{E}_{P_n^* P_n}(f,f) \geq c \inf_{f, Var_{\pi_n}(f)=1} \mathcal{E}_{P_n}(f,f) \tag{4.32}$$

where $\mathcal{E}_{P_n}(f,f) := \frac{1}{2}\sum_{x,y\in V_n} (f(x)-f(y))^2 P_n(x,y)\pi_n(x)$ and $P_n^*(x,y) := \frac{\pi_n(y)}{\pi_n(x)} P_n(y,x)$, and $\mathcal{E}_{P_n^* P_n}$ is defined analogously (note that π_n is invariant for P_n and $P_n^* P_n$).
This is clearly true for $\alpha > 0$: for all $x, y \in V_n$,

$$P_n^* P_n(x,y) \geq P_n^*(x,y) P_n(x,x) \geq \alpha P_n^*(x,y),$$

hence $\mathcal{E}_{P_n^* P_n}(f,f) \geq \alpha \mathcal{E}_{P_n}(f,f)$ for all f (note that $\mathcal{E}_{P_n}(f,f) = \mathcal{E}_{P_n^*}(f,f)$ for all f).

4.2 An example with a precise cutoff window: the NBRW on non-weighted graphs

4.2.1 Setting

Let $\mathcal{G}$ be a finite multigraph such that **A.1**, **A.2** and **A.4** hold. Suppose that for every $u \in V_{\mathcal{G}}$, all oriented edges going out of u have the same weight (this uniformity on the weights is also called the "non-weighted" case). In particular, **A.3** is satisfied. Denote $N = n|\overrightarrow{E}_{\mathcal{G}}|$ the cardinality of $\overrightarrow{E}_n$.

Let P (resp. P_n) be the transition matrix for the NBRW on $\overrightarrow{E}_{\mathcal{G}}$ (resp. $\overrightarrow{E}_n$), defined as in (4.9). It is irreducible by (4.8). We suppose that it is also aperiodic. One checks easily that P is bistochastic, so that its invariant measure is the uniform distribution. The same holds for P_n. We will denote $(Y_k)_{k\geq 0}$ a NBRW on $\overrightarrow{E}_n$ and $(\overline{Y}_k)_{k\geq 0}$ its projection on $\overrightarrow{E}_{\mathcal{G}}$. Note the following analogue of Lemma 4.1.7, with an identical proof:

Lemma 4.2.1 (Lemma 4.1.7 for a NBRW). *$(\overline{Y}_k)_{k\geq 0}$ is a NBRW on $\overrightarrow{E}_{\mathcal{G}}$.*

We denote ρ the pairing of the half-edges of $\overrightarrow{E}_n$. We will abusively identify each half-edge u with the oriented edge starting at u. We define $t_{n,u}(\varepsilon), t_n^{min}(\varepsilon)$ and $t_n^{max}(\varepsilon)$ as the ε-mixing times from $u \in \overrightarrow{E}_n$, in the worst and in the best case, as in (4.1).

Let

$$\mu := \frac{1}{|\overrightarrow{E}_{\mathcal{G}}|} \sum_{x\in\overrightarrow{E}_{\mathcal{G}}} \log\deg(x)$$

be the average log-degree of an element of $\overrightarrow{E}_{\mathcal{G}}$ (and thus of $\overrightarrow{E}_n$), where we set, for $\overrightarrow{e} \in \overrightarrow{E}_{\mathcal{G}}$: $\deg(x) := |\{y \in \overrightarrow{E}_{\mathcal{G}}, P(x,y) > 0\}|$. Define similarly $\deg(x)$ for $x \in \overrightarrow{E}_n$. Fix $\overrightarrow{e} \in \overrightarrow{E}$ and let

$$\sigma^2 := \frac{1}{|\overrightarrow{E}_{\mathcal{G}}|}\mathbb{E}\left[\left(\sum_{t=0}^{\tau_{\overrightarrow{e}}-1} \log(\deg(Z_t)) - \frac{1}{|\overrightarrow{E}_{\mathcal{G}}|}\mathbb{E}\left[\sum_{t=0}^{\tau_{\overrightarrow{e}}-1} \log(\deg(Z_t))\right]\tau_{\overrightarrow{e}}\right)^2\right],$$

where $(Z_t)_{t\geq 0}$ is a NBRW on $\overrightarrow{E}_{\mathcal{G}}$ with $Z_0 = \overrightarrow{e}$ a.s. and $\tau_{\overrightarrow{e}}$ is the first hitting time of $\overrightarrow{e}$ after 0. Note that $\deg(x) \geq 1$ for all $x \in \overrightarrow{E}_{\mathcal{G}}$ by **A.4** and that the maximal degree of an oriented edge is $\Delta - 1 \geq 2$ by **A.2** (recall that Δ is the maximal degree of a vertex in $\mathcal{G}$). Thus μ and σ are well-defined, and $\mu > 0$. By Lemma 4.1.8, σ^2 does not depend on the choice of $\overrightarrow{e}$. We make the additional assumption

A.5 $\sigma^2 > 0$.

Note that $\sigma^2 = 0$ iff $\tau_u^{-1} \sum_{t=0}^{\tau_{\overrightarrow{e}}-1} \log(\deg(Z_t))$ is constant a.s., so that many graphs $\mathcal{G}$ satisfy **A.5**. For instance, it is enough that there are two cycles containing $\overrightarrow{e}$ on which the average log-degree is not the same.

Without loss of generality, we impose $\sigma > 0$. For $n \geq 1$, let

$$t_n^* := \frac{\log n}{\mu} \text{ and } w_n^* := \sigma\sqrt{\frac{\log n}{\mu^3}}. \tag{4.33}$$

4.2.2 Results

The results that we obtain are similar to those of Section 4.1.

Proposition 4.2.2. *For any deterministic sequence $(\mathcal{G}_n)_{n\geq 1}$ of n-lifts of $\mathcal{G}$, and for any $\varepsilon \in (0,1)$,*

$$\liminf_{n\to+\infty} \frac{t_n^{min}(\varepsilon) - t_n^*}{w_n^*} \geq \Phi^{-1}(\varepsilon), \tag{4.34}$$

where Φ was defined in (4.3).

Note that this does not depend on the lift structure. For a typical lift (when n is large), this lower bound is attained:

Theorem 4.2.3. *Suppose $\mathcal{G}_n$ is a uniform random lift. Then for any $\varepsilon \in (0,1)$,*

$$\frac{t_n^{max}(\varepsilon) - t_n^*}{w_n^*} \xrightarrow{\mathbb{P}} \Phi^{-1}(\varepsilon). \tag{4.35}$$

Remark 4.2.4. *One can derive easily from our proofs below that if **A.5** does not hold (for instance if all vertices have the same degree), then for all $\varepsilon > 0$, $|t_n^{max}(\varepsilon) - t_n^*| + |t_n^{min}(\varepsilon) - t_n^*| = o(\sqrt{\log n})$. By comparison, in the case of a uniform regular graph, it has been shown in [101] that the window around the mixing time is of constant size.*

In Section 4.2.3, we give a short proof of Proposition 4.2.2 using a geometric growth argument, estimating the weight of a typical path by Lemma 4.1.8. This also gives a lower bound for Theorem 4.2.3. In Section 4.2.4, we show the upper bound. The proof is similar to the corresponding theorem in [27], for the NBRW on configuration models.

4.2.3 Proof of Proposition 4.2.2

Let $u \in \overrightarrow{E}_n$ and take $Y_0 = u$ a.s. For $k > 0$, let w_k be the **weight** of the path $(Y_0, \ldots, Y_k)$, i.e. the probability that $(Y_j)_{j\geq 0}$ follows this path on its first k steps. Note that w_k is also the weight of $(\overline{Y}_0, \ldots, \overline{Y}_k)$, since $\deg(\overline{Y}_j) = \deg(Y_j)$ for all j. From Lemma 4.2.1 and Lemma 4.1.8, we deduce that

$$\lim_{k\to+\infty} \mathbb{P}(\log w_k \in (-\mu k + a\sigma\sqrt{k}, -\mu k + b\sigma\sqrt{k})) = \Phi(a) - \Phi(b). \tag{4.36}$$

Fix $\varepsilon \in (0,1)$, take $\varepsilon' > \varepsilon$ and $a < \Phi^{-1}(\varepsilon')$. If $k = \lfloor t_n^* + a' w_n^* \rfloor$ for some constant $a' < a$, we have for n large enough by (4.33):

$$\begin{aligned} \mu k - a\sigma\sqrt{k} &\leq \mu\left(\frac{\log n}{\mu} + \mu a'\sigma\sqrt{\frac{\log n}{\mu^3}} - 1\right) - a\sigma\sqrt{\frac{\log n}{\mu} + a'\sigma\sqrt{\frac{\log n}{\mu^3}} - 1} \\ &\leq \log n - (a - a')\sigma\sqrt{\frac{\log n}{\mu}}. \end{aligned}$$

Thus (4.36) with b large enough such that $\Phi(a) - \Phi(b) > \varepsilon'$ yields

$$\mathbb{P}(w_k \geq \tau_n) > \varepsilon'$$

for n large enough, where $\tau_n := \exp(C\sqrt{\log n})/n$ for some arbitrary constant $C \in (0, \frac{(a-a')\sigma}{2\sqrt{\mu}})$. In other words, the NBRW has a probability at least ε' to stay on paths of weight at least τ_n during the first k steps. Since the probability that the walk took indeed a given path is equal to its weight, there are at most ε'/τ_n such paths. Thus, there exists $T_k \subseteq \overrightarrow{E}_n$ such that $|T_k| \leq \varepsilon'/\tau_n$ and $\mathbb{P}[Y_k \in T_k] \geq \varepsilon'$.

Thus for n large enough, the distance to equilibrium (defined as in (4.20)) satisfies

$$d_u(k) \geq \sum_{v\in T_k} \left(P_n^k(u,v) - \tfrac{1}{N}\right) \geq \varepsilon' - \tfrac{\varepsilon'}{N\tau_n} \geq \varepsilon'(1 - \exp(-C\sqrt{\log n})) \geq \varepsilon,$$

so that $k \leq t_{n,u}(\varepsilon)$. This is uniform in u, so that $\liminf_{n\to+\infty} \frac{t_n^{min}(\varepsilon)-t_n^*}{w_n^*} \geq a'$. Since Φ^{-1} is continuous, we can take a' arbitrarily close to $\Phi^{-1}(\varepsilon)$, and the proof is complete.

4.2.4 Proof of Theorem 4.2.3

The lower bound is a consequence of Proposition 4.2.2. We first suppose that the following Propositions 4.2.5 and 4.2.6 hold to establish the upper bound, before proving them.

We say that $u \in \overrightarrow{E}_n$ is a *root* if the ball $\overrightarrow{B}(u,r)$ made of the oriented paths of length r starting at u is a tree (i.e. it does not contain any cycle, even non-oriented), where

$$r := \lfloor \log\log n \rfloor.$$

Let $\mathcal{R}_n \subseteq \overrightarrow{E}_n$ be the set of roots. It is not true (even with high probability only) that all oriented edges are roots, however, most of them are and the NBRW quickly reaches a root (Proposition 4.2.5), independently of the starting point. And from any given root u, the probability of reaching any given root v after approximately $t_n^* + \lambda w_n^*$ steps is close to $(1-\Phi(\lambda))/N$ for any constant λ (Proposition 4.2.6). From there, we can go quickly to any other oriented edge, again due to Proposition 4.2.5 and to the fact that P is doubly stochastic.

Proposition 4.2.5. *It holds that*

$$\max_{u\in\overrightarrow{E}_n} P_n^r(u, \overrightarrow{E}_n \setminus \mathcal{R}_n) \xrightarrow{\mathbb{P}} 0. \tag{4.37}$$

Proposition 4.2.6. *For an even integer $t = t_n^* + \lambda w_n^* + o(w_n^*)$, it holds that*

$$\min_{x\in\mathcal{R}_n} \min_{y\in\mathcal{R}_n\setminus\overrightarrow{B}(x,r)} P_n^t(x,\rho(y)) \geq \frac{1-\Phi(\lambda)+o_{\mathbb{P}}(1)}{N}. \tag{4.38}$$

Now, consider t' even such that $t' = t_n^* + \lambda w_n^* + o(w_n^*)$, and let $t := t' - 2r$ (note that $t - t' = o(w_n^*)$). Since the pairing ρ of the half-edges is an involution and thus is bijective, we have for $u \in \overrightarrow{E}_n$:

$$d_u(t') = d_u(t+2r) = \sum_{v\in\overrightarrow{E}_n} \left(\frac{1}{N} - P_n^{t+2r}(u,\rho(v))\right)_+. \tag{4.39}$$

But for all $v \in \overrightarrow{E}_n$,

$$P_n^{t+2r}(u,\rho(v)) \geq \sum_{x\in\mathcal{R}_n} P_n^r(u,x) \sum_{y\in\mathcal{R}_n\setminus\overrightarrow{B}(x,r)} P_n^t(x,\rho(y))P_n^r(\rho(y),\rho(v)).$$

By Proposition 4.2.6, $P_n^t(x,\rho(y)) \geq \frac{1-\Phi(\lambda)+o_{\mathbb{P}}(1)}{N}$ for all $x \in \mathcal{R}_n$ and $y \in \mathcal{R}_n \setminus \overrightarrow{B}(x,r)$, and by symmetry, $P_n^r(\rho(y),\rho(v)) = P_n^r(v,y)$. Thus,

$$P_n^{t+2r}(u,\rho(v)) \geq \frac{1-\Phi(\lambda)+o_{\mathbb{P}}(1)}{N} \sum_{x\in\mathcal{R}_n} P_n^r(u,x) \left(\sum_{y\in\mathcal{R}_n} P_n^r(v,y) - \sum_{z\in\overrightarrow{B}(x,r)} P_n^r(v,z) \right),$$

where the $o_{\mathbb{P}}(1)$ does not depend on u or v. Let S_u be the set of oriented edges whose initial vertex is in $B(s_u,4r)$, where s_u is the initial vertex of u. If $v \in \overrightarrow{E}_n \setminus S_u$, for every $x \in V_n$ and $z \in \overrightarrow{B}(x,r)$, $P_n^r(u,x)P_n^r(v,z) = 0$, so that

$$P_n^{t+2r}(u,\rho(v)) \geq \frac{1-\Phi(\lambda)+o_{\mathbb{P}}(1)}{N} \left(1 - \max_{z\in\overrightarrow{E}_n} P_n^r(z,\overrightarrow{E}_n\setminus\mathcal{R}_n)\right)^2 \geq \frac{1-\Phi(\lambda)+o_{\mathbb{P}}(1)}{N}$$

by Proposition 4.2.6. Plugging this in (4.39), we get

$$d_u(t') \leq \Phi(\lambda) + o_{\mathbb{P}}(1) + \tfrac{1}{N}|S_u| \leq \Phi(\lambda) + o_{\mathbb{P}}(1),$$

since $|S_u| \leq 1+\Delta+\ldots+\Delta^{4r+1} \leq \sqrt{N}$ (recall that the maximum degree of a vertex is Δ). This is uniform in u, and we have

$$\max_{u\in\overrightarrow{E}_n} d_u(t') \leq \Phi(\lambda) + o_{\mathbb{P}}(1),$$

which yields the upper bound for Theorem 4.2.3.

Proof of Proposition 4.2.5

We say that $u \in \overrightarrow{E}_n$ is a *bulb* if $\overrightarrow{B}(u,2r)$ contains at most one cycle (non necessarily oriented). Let $\mathcal{B}_n \subseteq \overrightarrow{E}_n$ be the set of bulbs. We first prove that

$$\text{w.h.p., } \mathcal{B}_n = \overrightarrow{E}_n. \tag{4.40}$$

Proof of (4.40). Let $u \in \overrightarrow{E}_n$. Each oriented edge leads to at most $\Delta - 1$ others, hence $\overrightarrow{B}(u,2r)$ contains at most

$$1 + (\Delta-1) + \ldots + (\Delta-1)^{2r} \leq (\Delta-1)^{2r+1} =: K$$

oriented edges. Starting from u, we can explore its neighbourhood as described in Lemma 4.1.6, proceeding to no more than K pairings. Each pairing has a probability at most $K/(n-K)$ to create a cycle, since at every moment of the construction, for each type in $\overrightarrow{E}_{\mathcal{G}}$, there are at most K unpaired half-edges in $\overrightarrow{B}(u,2r)$ and at least $n-K$ unpaired half-edges in $\overrightarrow{E}_n$. Thus, if p is the probability that $\overrightarrow{B}(u,2r)$ contains at least two cycles, a union bound on the couple of pairings (i,j) where the first two cycles happen gives

$$p \leq \sum_{1\leq i<j\leq K} \left(\frac{K}{n-K}\right)^2 \leq \frac{K^4}{(n-K)^2} \leq \frac{CK^4}{N^2}$$

for some constant $C > 0$ since $K = o(n)$ and $N/n = |\overrightarrow{E}_\mathcal{G}|$. Note that $K^4 = o(N)$, so that there is a probability $o(1/N)$ that u is not a bulb. A union bound on $u \in \overrightarrow{E}_n$ yields the result. $\square$

Second, we prove that starting from a bulb, the probability that the NBRW is not on a root after r steps is $o(1)$, where the speed of convergence to 0 only depends on $\mathcal{G}$ (in particular, it neither depends on the choice of the initial bulb, nor on the lift structure).
Let $u \in \overrightarrow{E}_n$ and take $Y_0 = u$ a.s. If $\overrightarrow{B}(u, 2r)$ contains no cycle, then the NBRW is in $\mathcal{R}_n$ at each of the first r steps and we are done. Suppose that there is a unique cycle $\mathcal{C}$ in $\overrightarrow{B}(u, 2r)$. Let $C_t \subseteq \overrightarrow{E}_n$ be the set of states from which the probability to be in $\mathcal{R}_n$ after $r - t$ steps is less than 1. For $1 \leq t \leq r$, let p_t be the probability that $Y_t \in C_t$. It is enough to prove that

$$p_r \xrightarrow{\mathbb{P}} 0 \tag{4.41}$$

uniformly in u and the structure of $\overrightarrow{B}(u, 2r)$. Remark that from any $v \in \overrightarrow{B}(u, 2r)$, there is at most one transition that the NBRW can take in order to keep a chance to attain $\mathcal{C}$ before leaving $\overrightarrow{B}(u, 2r)$: else, there would be another cycle. There is one exception: when v is just outside $\mathcal{C}$ and points toward a vertex on it. Again, due to the uniqueness of $\mathcal{C}$, the walk can meet such a v only once. Moreover, if (Y_k) leaves this path to $\mathcal{C}$, then Y_r will surely be in $\mathcal{R}_n$. Therefore, for at least $r - 1$ steps,

$$p_{t+1} = p_t / \deg(Y_t).$$

Note that by **A.2** and **A.4**, there exists an oriented edge on $\overrightarrow{E}_\mathcal{G}$ that the NBRW $(\overline{Y}_k)$ cannot avoid oriented edges of degree at least 2 for more than $|\overrightarrow{E}_\mathcal{G}| - 1$ consecutive steps. Hence, the same is true for (Y_k), so that

$$p_r \leq 2^{-\lfloor r/|\overrightarrow{E}_\mathcal{G}| \rfloor}.$$

This yields (4.41) and the conclusion follows.

Proof of Proposition 4.2.6

Consider two distinct half-edges x, y. Recall the sequential construction of $\mathcal{G}_n$ from Lemma 4.1.6. Pairing half-edges one after another, we grow a couple of trees $\mathcal{T}_x$ and $\mathcal{T}_y$, starting from x and y respectively. These trees will have final height $t/2$ (recall that t is even), and are made of non-backtracking paths. We then estimate the probability that at least one half-edge of $\mathcal{T}_x$ is matched with one of $\mathcal{T}_y$, and this yields a lower bound of $P_n^t(x, y)$. This procedure is called "exposure process". It was already used in [101] and then [27], which we follow more closely. A noticeable difference with [27] is that one has to pay attention to the type of the leaves since one can match two half-edges if and only if their types correspond.

The construction

At the start, all half-edges are unpaired, and $\mathcal{T}_x$ and $\mathcal{T}_y$ are reduced to x and y respectively. A **move** consists of three steps:

Step 1: pick an unpaired half-edge $u \in \mathcal{T}$, where $\mathcal{T} := \mathcal{T}_x \cup \mathcal{T}_y$, of height less than $t/2$ and of maximal weight. The **height** of u in $\mathcal{T}$ is the height of its vertex in $\mathcal{T}_x$ or $\mathcal{T}_y$. The **weight** of u, denoted $\varpi(u)$, is the weight of the path from x or y to u, depending on whether $u \in \mathcal{T}_x$ or $u \in \mathcal{T}_y$. Choose any total order of the half-edges to break possible ties.

Step 2: match u with a random (uniform) unpaired half-edge v.

Step 3: add the vertex of v and all its half-edges in $\mathcal{T}$, if none of them was already in $\mathcal{T}$. This condition ensures that both $\mathcal{T}_x$ and $\mathcal{T}_y$ will not contain any cycle.

The construction ends when no half-edge remains available for Step 1.

After the construction, let $\mathcal{F}$ be the set of remaining unpaired half-edges in $\mathcal{T}$, called *leaves* of the trees. A leaf u is said to be *good* whenever $\varpi(u) > n^{-3/4}$. Let $\mathcal{L}_x$ and $\mathcal{L}_y$ be the sets of good leaves of $\mathcal{T}_x$ and $\mathcal{T}_y$. Denote $\mathcal{H}$ the set of the other leaves.

For any $e \in \overrightarrow{E}_{\mathcal{G}}$, let $\mathcal{L}_x^{\overrightarrow{e}}$ and $\mathcal{L}_y^{\overrightarrow{e}}$ be the subsets of good leaves of type $\overrightarrow{e}$ in $\mathcal{T}_x$ and type $\overrightarrow{e}^{-1}$ in $\mathcal{T}_y$ respectively. Define

$$\Upsilon_{\overrightarrow{e}}^{+} := \sum_{(u,v)\in\mathcal{L}_x^{\overrightarrow{e}}\times\mathcal{L}_y^{\overrightarrow{e}}} \varpi(u)\varpi(v)\mathbf{1}_{\varpi(u)\varpi(v)>\theta}, \tag{4.42}$$

$$\Upsilon_{\overrightarrow{e}}^{-} := \sum_{(u,v)\in\mathcal{L}_x^{\overrightarrow{e}}\times\mathcal{L}_y^{\overrightarrow{e}}} \varpi(u)\varpi(v)\mathbf{1}_{\varpi(u)\varpi(v)\leq\theta}, \tag{4.43}$$

and

$$\Upsilon_{\overrightarrow{e}} := \Upsilon_{\overrightarrow{e}}^{+} + \Upsilon_{\overrightarrow{e}}^{-}, \tag{4.44}$$

where $\theta := \frac{1}{n\log^2 n}$. We claim that it is enough to prove the following three Lemmas.

Lemma 4.2.7. *For all $\varepsilon > 0$,*

$$\mathbb{P}\left(nP_n^t(x,y) \leq \sum_{\overrightarrow{e}\in\overrightarrow{E}_{\mathcal{G}}} \Upsilon_{\overrightarrow{e}}^{-} - \varepsilon\right) = o\left(\frac{1}{n^2}\right). \tag{4.45}$$

Lemma 4.2.8. *For all $\varepsilon > 0$, and $\overrightarrow{e} \in \overrightarrow{E}_{\mathcal{G}}$,*

$$\mathbb{P}\left(\{\sum_{u\in\mathcal{L}_x^{\overrightarrow{e}}} \varpi(u) < \frac{1}{|\overrightarrow{E}_{\mathcal{G}}|} - \varepsilon\} \cap \{x,y\in\mathcal{R}_n\}\right) = o\left(\frac{1}{n^2}\right), \tag{4.46}$$

and the same holds for $\mathcal{L}_y^{\overrightarrow{e}}$ instead of $\mathcal{L}_x^{\overrightarrow{e}}$.

Lemma 4.2.9. *For all $\varepsilon > 0$,*

$$\mathbb{P}\left(\sum_{\overrightarrow{e}\in\overrightarrow{E}_{\mathcal{G}}} \Upsilon_{\overrightarrow{e}}^{+} \geq \frac{\Phi(\lambda)}{|\overrightarrow{E}_{\mathcal{G}}|} + \varepsilon\right) = o\left(\frac{1}{n^2}\right). \tag{4.47}$$

Fix indeed $\varepsilon > 0$, and suppose that the conclusions of these Lemmas hold. Note that the $o(1/n^2)$ is uniform over the pairs $x, y \in \mathcal{R}_n$ in every Lemma (since the events considered might only depend on the types of x and y, for which we have no more than $|\overrightarrow{E}_\mathcal{G}|$ choices). Hence, by a union bound, these conclusions hold w.h.p. for all pairs $x, y \in \mathcal{R}_n$. Then

$$\begin{aligned} P_n^t(x,y) &\geq n^{-1}\left(\sum_{\overrightarrow{e}\in\overrightarrow{E}_\mathcal{G}} \Upsilon^-_{\overrightarrow{e}} - \varepsilon\right) \\ &\geq n^{-1}\left(\sum_{\overrightarrow{e}\in\overrightarrow{E}_\mathcal{G}} \Upsilon^-_{\overrightarrow{e}} + \sum_{\overrightarrow{e}\in\overrightarrow{E}_\mathcal{G}} \Upsilon^+_{\overrightarrow{e}} - \left(\frac{\Phi(\lambda)}{|\overrightarrow{E}_\mathcal{G}|} + \varepsilon\right) - \varepsilon\right) \\ &\geq n^{-1}\left(\sum_{\overrightarrow{e}\in\overrightarrow{E}_\mathcal{G}} \Upsilon^-_{\overrightarrow{e}} + \sum_{\overrightarrow{e}\in\overrightarrow{E}_\mathcal{G}} \Upsilon^+_{\overrightarrow{e}}\right) - \frac{\Phi(\lambda)}{N} - \frac{2|\overrightarrow{E}_\mathcal{G}|\varepsilon}{N} \end{aligned}$$

by Lemmas 4.2.7 and 4.2.9. Moreover,

$$\begin{aligned} \sum_{\overrightarrow{e}\in\overrightarrow{E}_\mathcal{G}} \Upsilon^+_{\overrightarrow{e}} + \Upsilon^-_{\overrightarrow{e}} &= \sum_{\overrightarrow{e}\in\overrightarrow{E}_\mathcal{G}} \left(\sum_{u\in\mathcal{L}_x^{\overrightarrow{e}}} \varpi(u) \sum_{v\in\mathcal{L}_y^{\overrightarrow{e}}} \varpi(v)\right) \\ &\geq |\overrightarrow{E}_\mathcal{G}| \frac{1}{|\overrightarrow{E}_\mathcal{G}|^2} - \varepsilon \\ &\geq \frac{1}{|\overrightarrow{E}_\mathcal{G}|} - \varepsilon \end{aligned}$$

by Lemma 4.2.8. This yields

$$\begin{aligned} P_n^t(x,y) &\geq \frac{1}{N} - \frac{|\overrightarrow{E}_\mathcal{G}|\varepsilon}{N} - \frac{\Phi(\lambda)}{N} - \frac{2|\overrightarrow{E}_\mathcal{G}|\varepsilon}{N} \\ &\geq \frac{1-\Phi(\lambda)}{N} - \frac{3|\overrightarrow{E}_\mathcal{G}|\varepsilon}{N}. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the conclusion follows.
The remaining subsections are devoted to establishing these Lemmas.

Proof of Lemma 4.2.7

Fix $\varepsilon > 0$. We use Stein's method on concentration inequalities for exchangeable pairs, as presented in [52]. We need the weights to be not too large, and this is the reason why we needed to introduce $\Upsilon^-_{\overrightarrow{e}}$ and $\Upsilon^+_{\overrightarrow{e}}$. Since good leaves have height $t/2$, we have

$$P_n^t(x, \rho(y)) \geq \textstyle\sum_{\overrightarrow{e}\in\overrightarrow{E}_\mathcal{G}} W_{\overrightarrow{e}}$$

with

$$W_{\overrightarrow{e}} := \textstyle\sum_{u\in\mathcal{L}_x^{\overrightarrow{e}}, v\in\mathcal{L}_y^{\overrightarrow{e}}} \varpi(u)\varpi(v)\mathbf{1}_{u=\rho(v)}\mathbf{1}_{\varpi(u)\varpi(v)\leq\theta}.$$

Now, fix $\overrightarrow{e} \in \overrightarrow{E}_{\mathcal{G}}$. The sets of half-edges of type $\overrightarrow{e}$ (resp. $\overrightarrow{e}^{-1}$) that are still unpaired at the end of the construction have the same cardinality m. A uniform random matching between those sets shall take place in order to complete the construction of the random lift (due to Lemma 4.1.6). We apply Proposition 1.1 in [52] to the random variable $\widetilde{W}_{\overrightarrow{e}} := \frac{W_{\overrightarrow{e}}}{\theta}$: for any $a \geq 0$,

$$\mathbb{P}(|\widetilde{W}_{\overrightarrow{e}} - \mathbb{E}[\widetilde{W}_{\overrightarrow{e}}]| \geq a) \leq 2\exp\left(\frac{-a^2}{4\mathbb{E}[\widetilde{W}_{\overrightarrow{e}}] + 2a}\right),$$

where the expectation is taken with respect to the pairing. Note that $\mathbb{E}[\widetilde{W}_{\overrightarrow{e}}] = \frac{1}{m\theta}\Upsilon^{-}_{\overrightarrow{e}}$ and that $m = n + o(n)$ uniformly over all possibilities for the exposure process. Therefore, taking $a = \frac{\varepsilon}{2N\theta}$ in the previous inequality for all $\overrightarrow{e}$, we get

$$nP_n^t(x, \rho(y)) \geq \sum_{\overrightarrow{e} \in \overrightarrow{E}_{\mathcal{G}}} \Upsilon^{-}_{\overrightarrow{e}} - \varepsilon$$

with probability at least $1 - \exp(-\frac{\varepsilon^2}{Cn\theta})$ for some constant C (note that $\Upsilon^{-}_{\overrightarrow{e}}$ is bounded above by 1, so that $\mathbb{E}[\widetilde{W}_{\overrightarrow{e}}] \leq \frac{1}{m\theta}$). Since $\rho = 1/(n\log^2 n)$, this probability is $1 - o(1/n^2)$ and we are done.

Proof of Lemma 4.2.8

By symmetry, it is enough to prove the Lemma for $\mathcal{L}_x^{\overrightarrow{e}}$. Fix $\varepsilon > 0$. Take $Y_0 = x$ a.s. Then

$$\begin{aligned}\sum_{u \in \mathcal{L}_x^{\overrightarrow{e}}} \varpi(u) &= \mathbb{P}_{EP}(Y_{t/2} \text{ is of type } \overrightarrow{e} \text{ and is on } \mathcal{F} \setminus \mathcal{H}) \\ &\geq \mathbb{P}_{EP}(Y_{t/2} \text{ is of type } \overrightarrow{e}) - \mathbb{P}(Y_{t/2} \notin \mathcal{F} \setminus \mathcal{H}),\end{aligned}$$

where $\mathbb{P}_{EP}$ is the law of (Y_k) conditionally on the exposure process. Suppose that

$$\mathbb{P}(\{\sum_{u \in \mathcal{F}\setminus\mathcal{H}} \varpi(u) \leq 2 - \varepsilon/2\} \cap \{x, y \in \mathcal{R}_n\}) = o\left(\frac{1}{n^2}\right). \tag{4.48}$$

This entails that the event $\{\mathbb{P}_{EP}(Y_{t/2} \text{ is not located on } \mathcal{F} \setminus \mathcal{H}) > \varepsilon/2\} \cap \{x, y \in \mathcal{R}_n\}$ has probability $o(1/n^2)$. Since P is bistochastic and irreducible, $(\overline{Y}_k)$ has a unique stationary distribution, which is uniform. For n (and thus t) large enough,

$$\mathbb{P}_{EP}(Y_{t/2} \text{ is of type } \overrightarrow{e}) = \mathbb{P}_{EP}(\overline{Y}_{t/2} \text{ is of type } \overrightarrow{e}) > 1/|\overrightarrow{E}_{\mathcal{G}}| - \varepsilon/2,$$

for every realization of the exposure process. Hence,

$$\mathbb{P}\left(\{\sum_{u \in \mathcal{L}_x^{\overrightarrow{e}}} \varpi(u) < \frac{1}{|\overrightarrow{E}_{\mathcal{G}}|} - \varepsilon\} \cap \{x, y \in \mathcal{R}_n\}\right) = o(1/n^2).$$

Therefore, it only remains to establish (4.48).

Proof of (4.48). Take $\delta = \varepsilon/4$. We split the proof into two natural parts: first, we prove that

$$\mathbb{P}\left(\sum_{u\in\mathcal{H}} \varpi(u) \geq \delta\right) = o\left(\frac{1}{n^2}\right).$$

Second, we show that

$$\mathbb{P}(\{\sum_{u\in\mathcal{F}} \varpi(u) \leq 2-\delta\} \cap \{x, y \in \mathcal{R}_n\}) = o\left(\frac{1}{n^2}\right). \tag{4.49}$$

As for the first part, set $Y_0 = x$ a.s. Then

$$\sum_{u\in\mathcal{H}_x} \varpi(u) \leq \mathbb{P}\left(\prod_{j=0}^{t/2-1} \frac{1}{\deg(Y_j)} \leq n^{-3/4}\right),$$

where $\mathcal{H}_x := \mathcal{T}_x \cap \mathcal{H}$. By Lemma 4.1.8,

$$\mathbb{P}\left(\prod_{j=0}^{t/2-1} \frac{1}{\deg(Y_j)} \leq n^{-3/4}\right) = \mathbb{P}\left(\prod_{j=0}^{t/2-1} \frac{1}{\deg(\overline{Y}_j)} \leq n^{-3/4}\right) \leq \delta/2$$

for n large enough. We treat $\sum_{u\in\mathcal{H}_y} \varpi(u)$ the same way, and the conclusion follows.
It remains to prove (4.49), which we do by introducing a martingale of which we control the first and second moments.
Suppose that $x, y \in \mathcal{R}_n$. Let τ be the random number of moves in the exposure process, and $(\mathcal{F}_j)_{j\geq 0}$ be the natural filtration associated to this process: τ is a stopping time with respect to this filtration. Let U_j be the set of unpaired half-edges in $\mathcal{T}$ after j moves for $j \geq 0$, and $K_j := \sum_{u\in U_j} \varpi(u)$. Observe that the sequence $(K_j)_{j\geq 0}$ is non-increasing, and that we have $K_0 = K_r = 2$ since x and y are roots. Then, when we pick an unpaired half-edge $u \in \mathcal{T}$ and match it with another half-edge v, it might happen that v is also in $\mathcal{T}$. Thus, we lose $\varpi(u)$ and $\varpi(v)$. Formally, for $j \geq r+1$,

$$K_j = K_{j-1} - \mathbf{1}_{j\leq\tau}\mathbf{1}_{v_j\in U_{j-1}}\left(\varpi(u_j) + \varpi(v_j)\right), \tag{4.50}$$

where u_j is the unpaired half-edge picked at the j-th move and v_j is the half-edge to which it is matched. Remark that $\varpi(u_j)$ is $\mathcal{F}_{j-1}$-measurable. It holds:

$$\begin{aligned}\mathbb{E}\left[K_j - K_{j-1}|\mathcal{F}_{j-1}\right] &= -\mathbf{1}_{j\leq\tau}\left(\varpi(u_j)\mathbb{P}(v_j \in U_{j-1}|\mathcal{F}_{j-1}) + \sum_{v\in U_{j-1}} \varpi(v_j)\mathbb{P}(v_j = v|\mathcal{F}_{j-1})\right)\\ &\geq -\mathbf{1}_{j\leq\tau}\left(\frac{\varpi(u_j)|U_{j-1}| + K_{j-1}}{n-2j+1}\right).\end{aligned}$$

Indeed, $2j-2$ half-edges have been paired, so that there are still at least $n-2j+1$ unpaired half-edges (other than u_j) of each type. Hence, for every $v \in U_{j-1}$, $\mathbb{P}(v_j = v|\mathcal{F}_{j-1}) \leq \frac{1}{n-2j+1}$, which implies

$\mathbb{P}(v_j \in U_{j-1}|\mathcal{F}_{j-1}) \leq \frac{|U_{j-1}|}{n-2j+1}$ and $\sum_{v\in U_{j-1}} \varpi(v_j)\mathbb{P}(v_j = v|\mathcal{F}_{j-1}) \leq \frac{\sum_{v\in U_{j-1}}\varpi(v)}{n-2j+1} = \frac{K_{j-1}}{n-2j+1}$.

We also have

$$(K_j - K_{j-1})^2 = \mathbf{1}_{j\le\tau}\mathbf{1}_{v_j\in U_{j-1}}(\varpi(u_j)+\varpi(v_j))^2,$$

hence

$$\mathbb{E}\left[(K_j - K_{j-1})^2|\mathcal{F}_{k-1}\right] \le \mathbf{1}_{j\le\tau}\left(\frac{\varpi(u_j)^2|U_{j-1}| + 2\varpi(u_j)K_{j-1} + \sum_{v\in U_{j-1}}\varpi(v)^2}{n-2j+1}\right).$$

Since half-edges are selected in decreasing order of weight, for every $u \in U_{j-1}$, its parent has weight at least $\varpi(u_j)$, thus $\varpi(u) \ge \frac{\varpi(u_j)}{\Delta}$. But the total weight of the elements of U_{j-1} is $K_{j-1} \le 2$, so that $|U_{j-1}| \le \frac{2\Delta}{\varpi(u_j)}$. Note also that $\varpi(v)^2 \le \varpi(v)$ for any v. Hence

$$\mathbb{E}\left[K_j - K_{j-1}|\mathcal{F}_{k-1}\right] \ge -\mathbf{1}_{j\le\tau}\,\frac{2\Delta+2}{n-2j+1}, \quad \text{and}$$

$$\mathbb{E}\left[(K_j - K_{j-1})^2|\mathcal{F}_{k-1}\right] \le \mathbf{1}_{j\le\tau}\,\frac{2\Delta\varpi(u_j)+6}{n-2j+1}.$$

We sum those inequalities for $j = 1, \ldots, \tau$. Observe that the sum of the weights of all the half-edges $u_1, \ldots, u_\tau$ that were once selected in the exposure process is at most $t+2$, since there are $t/2+1$ generations on each tree $\mathcal{T}_x$, $\mathcal{T}_y$, the total mass on each generation being at most 1. Hence

$$\tau n^{-3/4} \le \sum_{j=1}^{\tau}\varpi(u_j) \le t+2.$$

Since $t = o(n^{1/8})$, we have $\tau = o(n)$ and we get for n large enough:

$$\sum_{k=1}^{\tau}\mathbb{E}\left[K_j - K_{j-1}|\mathcal{F}_{k-1}\right] \ge -3\Delta t n^{-1/4} \tag{4.51}$$

and

$$\sum_{k=1}^{\tau}\mathbb{E}\left[(K_j - K_{j-1})^2|\mathcal{F}_{k-1}\right] \le 3\Delta t n^{-1/4}. \tag{4.52}$$

Denote α and β the RHS of (4.51) and (4.52) respectively. Fix $\gamma > 0$ and define the martingale M_γ by setting $M_\gamma(0) = 0$ and

$$M_\gamma(k) := \sum_{j=1}^{k}(K_{j-1} - K_j)\wedge\gamma - \mathbb{E}\left[(K_{j-1} - K_j)\wedge\gamma|\mathcal{F}_{j-1}\right] \tag{4.53}$$

for $k \in \mathbb{N}$. It is straightforward that $\mathbb{P}(|M_\gamma(k) - M_\gamma(k-1)| \le \gamma) = 1$, and we may apply Proposition 2.1 (p.5) in [77] with $a = 9\gamma$, $b = \beta$ and $K = \gamma$ to get:

$$\mathbb{P}(M_\gamma(\tau) \ge 9\gamma) \le e^9\left(\frac{\beta}{\beta+9\gamma^2}\right)^{9+\beta/\gamma^2} \le C\beta^9$$

for some constant $C > 0$ uniquely depending on γ. And $\beta = O(n^{-9/40})$ by definition of t in Proposition 4.2.6 and by (4.52), so that

$$\mathbb{P}(M_\gamma(\tau) \ge 9\gamma) = o\left(\frac{1}{n^2}\right). \tag{4.54}$$

To conclude, it is enough to show that for n large enough,

$$\max_{j\le\tau} |K_{j-1} - K_j| \le \gamma \text{ almost surely.} \tag{4.55}$$

Indeed, on $\{\max_{j\le\tau} |K_{j-1} - K_j| \le \gamma\}$, $K_\tau \ge K_0 - M_\gamma(\tau) + \alpha$. Moreover, $K_0 = 2$ and $\alpha = o(1)$. Together with (4.54), where we can take $\gamma > 0$ arbitrarily small, this yields the result.

The irreducibility of P and the existence of a half-edge with degree at least $\Delta - 1 \ge 2$ imply that every non-backtracking path on $\mathcal{G}$ meets an oriented edge with degree at least 2 every $|\overrightarrow{E}_\mathcal{G}|$ steps at most. The same holds on the balls of radius r around x and y, which are included in $\mathcal{T}_x$ and $\mathcal{T}_y$ respectively since $x, y \in \mathcal{R}_n$. This divides the weight of each path in $\mathcal{T}$ by at least 2 after at most $|\overrightarrow{E}_\mathcal{G}|$ steps. Hence, for n large enough, for every $j \ge r$, $\max(\varpi(u_j), \varpi(v_j)) < \gamma/2$ almost surely, so that $\max_{j\le\tau} |K_{j-1} - K_j| \le \gamma$ (recall that $K_0 = \ldots = K_r = 2$). This yields (4.55). □

Proof of Lemma 4.2.9

We proceed as in the first part of the proof of (4.48). Fix $\varepsilon > 0$ and $\overrightarrow{e} \in \overrightarrow{E}_\mathcal{G}$. Let $(Y_j)_{j\ge1}$ and $(Z_j)_{j\ge1}$ be independent NBRWs starting at x and y respectively. Let $\mathcal{E}$ be the event that $Y_{t/2}$ is of type $\overrightarrow{e}$, that $Z_{t/2}$ is of type $\overrightarrow{e}^{-1}$, and that

$$\prod_{j=1}^{t/2} \frac{1}{\deg(Y_j)\deg(Z_j)} \ge \theta.$$

Clearly, it holds that

$$\Upsilon^+_{\overrightarrow{e}} \le \mathbb{P}(\mathcal{E}).$$

Denote $(\overline{Z}_k)_{k\ge0}$ the projection of $(Z_k)_{k\ge0}$ on $\overrightarrow{E}_\mathcal{G}$, which is a NBRW by Lemma **??**. Note that Y_k and $\overline{Y}_k$ are of the same type for all k, and similarly for (Z_k), so that

$$\mathbb{P}\left(\prod_{j=1}^{t/2-r} \frac{1}{\deg(Y_j)\deg(Z_j)} \ge \theta\right) = \mathbb{P}\left(\prod_{j=1}^{t/2-r} \frac{1}{\deg(\overline{Y}_j)\deg(\overline{Z}_j)} \ge \theta\right),$$

where we recall that $r = \log\log n$. The proof of Lemma 4.1.8 in [54], which relies on an excursion decomposition of the trajectory of the random walk, allows the following slight modification: in the random walk $(X_t)_{0\le t\le n}$, we can replace $(X_t)_{n/2\le t\le n}$ by a random walk $(X'_t)_{0\le t\le n/2+1}$ starting at an arbitrary state in Ω. We apply this version of the Lemma by concatenating the trajectories of $(\overline{Y}_j)_{1\le j\le t/2-r}$ and $(\overline{Z}_j)_{1\le j\le t/2-r}$.

$$\begin{aligned}
\mathbb{P}\left(\prod_{j=1}^{t/2-r} \frac{1}{\deg(\overline{Y}_j)\deg(\overline{Z}_j)} \ge \theta\right) &= \mathbb{P}\left(\sum_{j=1}^{t/2-r} \log(\deg(\overline{Y}_j)) + \log(\deg(\overline{Z}_j)) \le -\log\theta\right) \\
=&\mathbb{P}\left(\sum_{j=1}^{t/2-r} \log(\deg(\overline{Y}_j)) + \log(\deg(\overline{Z}_j)) \le \mu(t-2r) + \lambda\sqrt{t-2r} + o(\sqrt{t-2r})\right) \\
=&\Phi(\lambda) + o(1).
\end{aligned}$$

Then, uniformly on $\overline{Y}_{t/2-r}$, $\overline{Y}_t$ is of type $\overrightarrow{e}$ with probability $\frac{1}{|\overrightarrow{E}_\mathcal{G}|} + o(1)$ since the stationary distribution of the NBRW on $\overrightarrow{E}_\mathcal{G}$ is uniform, and the same holds for $\overline{Z}_t$ (independently of $\overline{Y}_t$). Hence

$$\Upsilon^+_{\overrightarrow{e}} \leq \frac{\Phi(\lambda)+\varepsilon}{|\overrightarrow{E}_\mathcal{G}|^2},$$

and we obtain the desired result by summing on $\overrightarrow{e} \in \overrightarrow{E}_\mathcal{G}$.

4.3 Diameter of random lifts

In this section, we compute the oriented diameter D_n of $\mathcal{G}_n$. For $n \geq 1$, we set

$$D_n := \max_{x,y \in V_n} \min\{k \geq 0, y \in \overrightarrow{B}(x,k)\}. \tag{4.56}$$

We make the assumptions **A.1**, **A.2** and **A.4** on $\mathcal{G}$. Note that the distribution of D_n only depends on n and on which oriented edges of $\mathcal{G}$ have a positive weight. To compute the non-oriented diameter, it is enough to give a positive weight to both orientations of every edge, hence our setting already includes this case.

Let $A_\mathcal{G}$ be the **oriented adjacency matrix** of $\mathcal{G}$: it is a matrix indexed by $\overrightarrow{E}^+_\mathcal{G} \times \overrightarrow{E}^+_\mathcal{G}$ where $\overrightarrow{E}^+_\mathcal{G} \subseteq \overrightarrow{E}_\mathcal{G}$ is the set of oriented edges with a positive weight, and for every $(\overrightarrow{e}, \overrightarrow{e}') \in (\overrightarrow{E}^+_\mathcal{G})^2$,

$$A_\mathcal{G}(\overrightarrow{e}, \overrightarrow{e}') = \mathbf{1}_{\{\overrightarrow{e}^{-1} \text{ and } \overrightarrow{e}' \text{ start at the same vertex, and } \overrightarrow{e}' \neq \overrightarrow{e}^{-1}\}}.$$

Note that $A_\mathcal{G}$ is irreducible by (4.8). Let D be the Perron-Frobenius eigenvalue of $A_\mathcal{G}$, i.e. its largest eigenvalue, which is positive and simple by the Perron-Frobenius theorem. Then

Theorem 4.3.1. *As $n \to +\infty$,*

$$\frac{D_n}{\log n} \xrightarrow{\mathbb{P}} \log^{-1} D.$$

Let us give a brief intuition on this result. Since $\mathcal{G}_n$ is an expander (Proposition 4.1.25) with bounded degrees, its diameter should be of order $\log n$. The constant $\log^{-1} D$ comes from the fact that D is the growth rate of the universal cover, and that there are too few cycles in $\mathcal{G}_n$ to reduce this rate.

Remark 4.3.2. *Using the same core/branches decomposition than in Section 4.1.5, we see easily that the result holds without **A.4**, and that $A_\mathcal{G}$ and $A_{c(\mathcal{G})}$ have the same Perron-Frobenius eigenvalue.*

4.3.1 The growth rate of the universal cover

As in Section 4.1.3, root $\mathcal{T}_\mathcal{G}$ at some vertex $\circ$. Denote $\widetilde{B}(\circ, R)$ the oriented ball centred at $\circ$ of radius R, i.e. it consists of all oriented paths of length R started at $\circ$.

Lemma 4.3.3.
$$\frac{\log(|\partial \widetilde{B}(\circ, R)|)}{R} \underset{R\to+\infty}{\to} \log^{-1} D.$$

Naturally, this result is independent of the choice of $\circ$.

Proof. For $R \geq 1$, let $a_R := |\partial \widetilde{B}(\circ, R)|$, and for every $\overrightarrow{e} \in \overrightarrow{E}_{\mathcal{G}}$, let $a_R(\overrightarrow{e})$ be the number of vertices x in $|\partial \widetilde{B}(\circ, R)|$ such that if y is the parent of x, then (y, x) has type $\overrightarrow{e}$. Denote $\overrightarrow{a}_R$ the vector $(a_R(\overrightarrow{e}))_{\overrightarrow{e}\in \overrightarrow{E}_{\mathcal{G}}}$. Then for every $R \geq 1$, $\overrightarrow{a}_{R+1} = A_{\mathcal{G}} \overrightarrow{a}_R$ so that by an immediate induction:
$$\overrightarrow{a}_R = A_{\mathcal{G}}^{R-1} \overrightarrow{a}_1.$$

D is the largest eigenvalue of $A_{\mathcal{G}}$ and has multiplicity 1. Moreover, by **A.2**, $|\widetilde{B}(\circ, R)|$ grows exponentially with R so that $D > 1$. Therefore, there exist constants $c, c' > 0$, uniquely depending on $\mathcal{G}$, such that for every $R \geq 1$,
$$a_R = \textstyle\sum_{\overrightarrow{e}\in \overrightarrow{E}_{\mathcal{G}}} a_R(\overrightarrow{e}) \in [cD^R, c'D^R].$$

The conclusion follows. □

4.3.2 Proof of Theorem 4.3.1

Lemma 4.3.3 and the fact that $|\widetilde{B}(x, R)| \leq |\widetilde{B}(\circ, R)|$ for every x of the type of $\circ$ and every $R \geq 1$ imply that
$$\liminf_{n\to+\infty} \frac{D_n}{\log n} \underset{a.s.}{\geq} \log^{-1} D.$$

It remains to prove that this lower bound is optimal. We proceed in three steps. Fix $\varepsilon > 0$. First, we show that w.h.p., for every $x \in V_n$, the ball $\widetilde{B}(x, R_n)$ for $R_n := \lfloor (1/2-\varepsilon)\log^{-1} D \log n \rfloor$ has no more than $\lfloor 2/\varepsilon \rfloor + 1$ cycles, so that its successive generations grow exponentially at rate $\log D$, as $\widetilde{B}(\circ, R)$ when $R \to +\infty$. Second, for $y \notin \widetilde{B}(x, R_n)$, we build similarly the "adjoint" ball $\widetilde{B}^*(y, R_n)$, made of the oriented paths reaching y. The corresponding adjacency matrix is the transpose of $A_{\mathcal{G}}$, so that it has the same Perron-Frobenius eigenvalue D. Third, we use the expansion to connect $\widetilde{B}(x, R_n)$ to $\widetilde{B}^*(y, R_n)$ in $K_{\mathcal{G}}\varepsilon \log n$ steps, $K_{\mathcal{G}}$ being a constant uniquely depending on $\mathcal{G}$.

Step 1 Let $R_n := \lfloor (1/2 - \varepsilon)\log^{-1} D \log n \rfloor$. We claim that w.h.p., $\mathcal{G}_n$ is such that for every $x \in V_n$,
$$n^{1/2-2\varepsilon} \leq |\partial \widetilde{B}(x, R_n)| \leq n^{1/2-\varepsilon/2}. \tag{4.57}$$

Proof. Let $k_\varepsilon = \lfloor 2/\varepsilon \rfloor + 1$. We first show that w.h.p., for every $x \in V_n$,
$$\widetilde{B}(x, R_n) \text{ contains at most } k_\varepsilon \text{ (non-oriented) cycles.} \tag{4.58}$$

Let $x \in V_n$. Reveal $\widetilde{B}(x, R_n)$ in a breadth-first way, by successive pairings of half-edges. Since $\widetilde{B}(x, R_n)$ contains no more vertices of each type than $\widetilde{B}(\circ, R_n)$, the total number s of pairings satisfies

$$s \leq \Delta|\widetilde{B}(\circ, R_n)| \leq n^{1/2-\varepsilon/2}$$

for n large enough, by Lemma 4.3.3. Hence, the probability that more than k_ε cycles appear during those pairings is less than

$$\left(\frac{n^{1/2-\varepsilon/2}}{n-n^{1/2-\varepsilon/2}}\right)^{k_\varepsilon}\binom{\lfloor n^{1/2-\varepsilon/2}\rfloor}{k_\varepsilon} \leq 2^{k_\varepsilon} n^{(1/2-\varepsilon/2)2k_\varepsilon} n^{-k_\varepsilon} \leq n^{-3/2}.$$

Note indeed that at each step of the construction, at most $n^{1/2-\varepsilon/2}$ unpaired half-edges have been discovered and at least $n - n^{1/2-\varepsilon/2}$ half-edges of each type have not been paired. This yields (4.58).

Then, let $\mathcal{T}_x$ be a spanning tree of $\widetilde{B}(x, R_n)$: when building $\widetilde{B}(x, R_n)$, do not include in $\mathcal{T}_x$ the edges that close a cycle. For $k = 1$ to R_n and every $\overrightarrow{e} \in \overrightarrow{E}$, let $a_{k,n}(\overrightarrow{e})$ be the number of oriented edges of $\mathcal{T}_x$ from the k-th to the $(k+1)$-th generation, and $\overrightarrow{a}_{k,n}$ the vector $(a_{k,n}(\overrightarrow{e}))_{\overrightarrow{e}\in\overrightarrow{E}_\mathcal{G}}$. It satisfies the same recursive equation as $(\overrightarrow{a}_R)_{R\geq 0}$ in Lemma 4.3.3, with at most k_ε "-1" distributed arbitrarily to some $a_{k,n}(\overrightarrow{e})$, for $k \geq 1$ and $\overrightarrow{e} \in \overrightarrow{E}_\mathcal{G}$. Unless $a_{k,n} := \sum_{\overrightarrow{e}\in\overrightarrow{E}_\mathcal{G}} a_{k,n}(\overrightarrow{e}) = 0$ for some $k \geq 1$ (w.h.p., this happens for no $x \in V_n$, by Proposition 4.1.25), a bounded number of "-1"'s does not prevent the exponential growth of $a_{k,n}$ at rate $\log D$. Indeed, in the first $\log\log n$ generations, there are at least $(2k_\varepsilon)^{-1}\log\log n$ consecutive generations without a -1, during which $a_{k,n}(\overrightarrow{e})$ becomes larger than $\log\log n$ for every $\overrightarrow{e}$ (for n large enough). Then, each subsequent -1 divides the size of $a_{k,\varepsilon}(\overrightarrow{e})$ by a factor less than 2. Therefore, as in Lemma 4.3.3, there exist constants $c, c' > 0$ such that for n large enough, $a_{R_n,n} \in [cD^{R_n}, c'D^{R_n}]$. The conclusion follows. □

Step 2 Let $y \in V_n \setminus \widetilde{B}(x, R_n)$ and $v^*(y, R_n)$ be the ball centred at y made of the oriented paths of length R_n arriving at y. Define similarly $\widetilde{B}^*(\circ, R_n)$. We can readily adapt Lemma 4.3.3 and thus (4.57), $A_\mathcal{G}$ being replaced by its transpose $A^*_\mathcal{G}$, the latter having the same Perron-Frobenius eigenvalue.

Step 3 If $\widetilde{B}^*(y, R_n)$ intersects $\widetilde{B}(x, R_n)$, then there exists an oriented path from x to y of length at most $2R_n \leq \frac{\log n}{\log D}$. Suppose that this is not the case. We expand $\widetilde{B}(x, R_n)$ until we hit $\widetilde{B}^*(y, R_n)$.

Let $K_\mathcal{G} := L\pi_{min}/2$, where we recall that π_{min} is the smallest weight of the invariant probability distribution π of the SRW on $\mathcal{G}$, and where L was defined in (4.25). By Proposition 4.1.25, w.h.p. $\mathcal{G}_n$ is such that

$$\text{for every } x \in V_n \text{ and for } i \geq 1,\ |\widetilde{B}(x, R_n + i)| \geq (1 + K_\mathcal{G})^i |\widetilde{B}(x, R_n)|, \tag{4.59}$$

as long as $|\widetilde{B}(x, R_n + i)| \leq n\pi_{min}/2$. Let

$$R'_n := 10\varepsilon \log^{-1}(1 + K_\mathcal{G}) \log n.$$

Starting from $\widetilde{B}(x, R_n)$, reveal $\widetilde{B}(x, R_n + R'_n - |\overrightarrow{E}_{\mathcal{G}}|)$ in a breadth-first way. If we hit $\widetilde{B}^*(y, R_n)$, then there is an oriented path from x to y of length at most $2R_n + R'_n$. Else,

$$|\partial \widetilde{B}(x, R_n + R'_n - |\overrightarrow{E}_{\mathcal{G}}|)| \geq n^{1/2+6\varepsilon}$$

by (4.59). Then, by Step 1, Step 2 and the pigeon-hole principle, there exist $\overrightarrow{e}, \overrightarrow{e}' \in \overrightarrow{E}$ such that

- there are at least $|\partial \widetilde{B}^*(y, R_n)|/|\overrightarrow{E}_{\mathcal{G}}| \geq n^{1/2-3\varepsilon}$ vertices of $\partial \widetilde{B}^*(y, R_n)$ that are the initial vertex of an oriented edge of type $\overrightarrow{e}$ (and that edge does not belong to $\widetilde{B}^*(y, R_n)$),
- there are at least $|\partial \widetilde{B}(x, R_n + R'_n - |\overrightarrow{E}_{\mathcal{G}}|)|/|\overrightarrow{E}_{\mathcal{G}}| \geq n^{1/2+5\varepsilon}$ vertices of $\partial \widetilde{B}(x, R_n + R'_n - |\overrightarrow{E}_{\mathcal{G}}|)$ that are the initial vertex of an oriented edge of type $\overrightarrow{e}'$ (and that edge does not belong to $\widetilde{B}(x, R_n + R'_n - |\overrightarrow{E}_{\mathcal{G}}|)$).

Denote respectively A_y and A_x these sets of vertices. Now, we build a path of length at most $|\overrightarrow{E}_{\mathcal{G}}|$ from A_x to A_y via a binomial argument. Let $z_1, z_2, \ldots$ be the vertices of A_x, listed in an arbitrary order. For $i = 1, 2, \ldots$ successively, if no oriented path has been discovered from some $z_j, j < i$ to A_y, reveal one path from z_i to some oriented edge of type $\overrightarrow{e}$ (such a path exists by (4.8) applied to $\overrightarrow{e}_a = \overrightarrow{e}'$ and $\overrightarrow{e}_b = \overrightarrow{e}$). The probability that this path ends on A_y is at least $n^{1/2-3\varepsilon}/n$. Therefore, the probability that for some $i \geq 1$, one of these oriented paths lands on A_y is at least

$$1 - \mathbb{P}(Z = 0), \text{ where } Z \sim \text{Bin}(\lfloor n^{1/2+5\varepsilon} \rfloor, n^{-1/2-3\varepsilon}).$$

But for n large enough,

$$\mathbb{P}(Z = 0) = (1 - n^{-1/2-3\varepsilon})^{\lfloor n^{1/2+5\varepsilon} \rfloor} \leq \exp(\lfloor n^{1/2+5\varepsilon} \rfloor \log(1 - n^{-1/2-3\varepsilon})) \leq \exp(-n^{\varepsilon}) \leq n^{-3}.$$

By a union bound on $x, y \in V_n$, the probability that such a connection does not exist for some $x, y \in V_n$ is $o(1)$.

All in all, we have shown that w.h.p., $\mathcal{G}_n$ is such that for every $x, y \in V_n$, there is an oriented path of length at most

$$2R_n + R'_n \leq \log^{-1} D \log n + 10\varepsilon \log^{-1}(1 + K_{\mathcal{G}}) \log n$$

from x to y. Since $\varepsilon > 0$ can be taken arbitrarily small, this concludes the proof of the upper bound and hence of Theorem 4.3.1.

Bibliography

[1] Angelo Abächerli. Local picture and level-set percolation of the gaussian free field on a large discrete torus. *Stochastic Processes and their Applications*, 129(9):3527 – 3546, 2019.

[2] Angelo Abächerli and Alain-Sol Sznitman. Level-set percolation for the gaussian free field on a transient tree. *Ann. Inst. H. Poincaré Probab. Statist.*, 54(1):173–201, 02 2018.

[3] Angelo Abächerli and Jiří Černý. Level-set percolation of the Gaussian free field on regular graphs I: regular trees. *Electronic Journal of Probability*, 25(none):1 – 24, 2020.

[4] Angelo Abächerli and Jiří Černý. Level-set percolation of the Gaussian free field on regular graphs II: finite expanders. *Electronic Journal of Probability*, 25(none):1 – 39, 2020.

[5] Louigi Addario-Berry, Nicolas Broutin, and Christina Goldschmidt. Critical random graphs: limiting constructions and distributional properties. *Electron. J. Probab.*, 15:741–775, paper no. 25 (electronic), 2010.

[6] Louigi Addario-Berry, Nicolas Broutin, and Christina Goldschmidt. The continuum limit of critical random graphs. *Probab. Theory Related Fields*, 152(3–4):367–406, 2012.

[7] Louigi Addario-Berry, Nicolas Broutin, Christina Goldschmidt, and Grégory Miermont. The scaling limit of the minimum spanning tree of the complete graph. *Ann. Probab.*, 45(5):739–805, 2017.

[8] Louigi Addario-Berry and Simon Griffiths. The spectrum of random lifts. `https://arxiv.org/abs/1012.4097`, 2010.

[9] David Aldous. Random walks on finite groups and rapidly mixing markov chains. In Jacques Azéma and Marc Yor, editors, *Séminaire de Probabilités XVII 1981/82*, pages 243–297, Berlin, Heidelberg, 1983. Springer Berlin Heidelberg.

[10] David Aldous. The continuum random tree. I. *Ann. Probab.*, 19(1):1–28, 1991.

[11] David Aldous. The continuum random tree. II. An overview. In *Stochastic analysis (Durham, 1990)*, volume 167 of *London Math. Soc. Lecture Note Ser.*, pages 23–70. Cambridge Univ. Press, Cambridge, 1991.

[12] David Aldous. The continuum random tree. III. *Ann. Probab.*, 21(1):248–289, 1993.

[13] David Aldous. Brownian excursions, critical random graphs and the multiplicative coalescent. *Ann. Probab.*, 25(2):812–854, 1997.

[14] David Aldous and Persi Diaconis. Shuffling cards and stopping times. *Amer. Math. Monthly*, 93(5):333–348, 1986.

[15] David Aldous and Vlada Limic. The entrance boundary of the multiplicative coalescent. *Electron. J. Probab.*, 3:No. 3, 59 pages, 1998.

[16] David Aldous and Jim Pitman. Inhomogeneous continuum random trees and the entrance boundary of the additive coalescent. *Probab. Theory Related Fields*, 118(4):455–482, 2000.

[17] N Alon and V.D. Milman. Eigenvalues, expanders and superconcentrators, 11 1984.

[18] Noga Alon, Itai Benjamini, and Alan Stacey. Percolation on finite graphs and isoperimetric inequalities. *Ann. Probab.*, 32(3):1727–1745, 07 2004.

[19] Noga Alon and Joel H. Spencer. *The probabilistic method*. Wiley Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., Hoboken, NJ, fourth edition, 2016.

[20] Alon Amit and Nathan Linial. Random lifts of graphs: edge expansion. *Combin. Probab. Comput.*, 15(3):317–332, 2006.

[21] Alon Amit, Nathan Linial, and Jiří Matoušek. Random lifts of graphs: independence and chromatic number. *Random Structures Algorithms*, 20(1):1–22, 2002.

[22] Alon Amit, Nathan Linial, Jiří Matoušek, and Eyal Rozenman. Random lifts of graphs. In *Proceedings of the Twelfth Annual ACM-SIAM Symposium on Discrete Algorithms (Washington, DC, 2001)*, pages 883–894. SIAM, Philadelphia, PA, 2001.

[23] Richard Arratia, A. D. Barbour, and Simon Tavaré. *Logarithmic combinatorial structures: a probabilistic approach*. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2003.

[24] Luca Avena, Hakan Güldaş, Remco van der Hofstad, and Frank den Hollander. Random walks on dynamic configuration models:a trichotomy. *Stochastic Processes and their Applications*, 129(9):3360–3375, 2019.

[25] E. Barouch and G. M. Kaufman. Probabilistic modelling of oil and gas discovery. In F. S. Roberts, editor, *Energy: Mathematics and Models*, pages 133–150. SIAM, 1976.

[26] Anna Ben-Hamou, Eyal Lubetzky, and Yuval Peres. Comparing mixing times on sparse random graphs. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1734–1740. SIAM, Philadelphia, PA, 2018.

[27] Anna Ben-Hamou and Justin Salez. Cutoff for nonbacktracking random walks on sparse random graphs. *Ann. Probab.*, 45(3):1752–1770, 2017.

[28] Edward A. Bender and E. Rodney Canfield. The asymptotic number of labeled graphs with given degree sequences. *J. Combinatorial Theory Ser. A*, 24(3):296–307, 1978.

[29] Nathanaël Berestycki, Eyal Lubetzky, Yuval Peres, and Allan Sly. Random walks on the random graph. *Ann. Probab.*, 46(1):456–490, 01 2018.

[30] Jean Bertoin. Factorizing Laplace exponents in a spectrally positive Lévy process. *Stochastic Process. Appl.*, 42(2):307–313, 1992.

[31] Jean Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[32] Shankar Bhamidi, Nicolas Broutin, Sanchayan Sen, and Xuan Wang. Scaling limits of random graph models at criticality: Universality and the basin of attraction of the Erdős-Rényi random graph. `https://arxiv.org/abs/1411.3417`, 2014.

[33] Shankar Bhamidi, Amarjit Budhiraja, and Xuan Wang. The augmented multiplicative coalescent, bounded size rules and critical dynamics of random graphs. *Probab. Theory Related Fields*, 160(3-4):733–796, 2014.

[34] Shankar Bhamidi, Amarjit Budhiraja, and Xuan Wang. Bounded-size rules: the barely subcritical regime. *Combin. Probab. Comput.*, 23(4):505–538, 2014.

[35] Shankar Bhamidi, Souvik Dhara, Remco van der Hofstad, and Sanchayan Sen. Universality for critical heavy-tailed network models: Gromov–Hausdorff–Prohorov convergence. in preparation.

[36] Shankar Bhamidi, Souvik Dhara, Remco van der Hofstad, and Sanchayan Sen. Universality for critical heavy-tailed network models: Metric structure of maximal components. *Electronic Journal of Probability*, 25(none):1 – 57, 2020.

[37] Shankar Bhamidi and Sanchayan Sen. Geometry of the vacant set left by random walk on random graphs, wright's constants, and critical random graphs with prescribed degrees. *Random Structures & Algorithms*, 56(3):676–721, 2020.

[38] Shankar Bhamidi, Sanchayan Sen, and Xuan Wang. Continuum limit of inhomogenous random graphs. *Probab. Theory Related Fields*, 169(1-2):565–641, 2017.

[39] Shankar Bhamidi, Remco van der Hofstad, and Sanchayan Sen. The multiplicative coalescent, inhomogeneous continuum random trees, and new universality classes for critical random graphs. *Probab. Theory Related Fields*, 170(1-2):387–474, 2018.

[40] Shankar Bhamidi, Remco van der Hofstad, and Johan S. H. van Leeuwaarden. Scaling limits for critical inhomogeneous random graphs with finite third moments. *Electron. J. Probab.*, 15:no. 54, 1682–1703, 2010.

[41] Shankar Bhamidi, Remco van der Hofstad, and Johan S. H. van Leeuwaarden. Novel scaling limits for critical inhomogeneous random graphs. *Ann. Probab.*, 40(6):2299–2361, 2012.

[42] Tom Bohman and Alan Frieze. Avoiding a giant component. *Random Structures Algorithms*, 19(1):75–85, 2001.

[43] Béla Bollobás. A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. *European J. Combin.*, 1(4):311–316, 1980.

[44] Béla Bollobás. The isoperimetric number of random regular graphs. *European Journal of Combinatorics*, 9(3):241 – 244, 1988.

[45] C. Bordenave. A new proof of Friedman's second eigenvalue Theorem and its extension to random lifts. `https://arxiv.org/abs/1502.04482`, 2015.

[46] Charles Bordenave and Benoît Collins. Eigenvalues of random lifts and polynomials of random permutation matrices. *Annals of Mathematics*, 190(3):811–875, 2019.

[47] Charles Bordenave and Hubert Lacoin. Cutoff at the entropic time for random walks on covered expander graphs. *Journal of the Institute of Mathematics of Jussieu*, page 1–46, 2021.

[48] Jean Bricmont, Joel Lebowitz, and Christian Maes. Percolation in strongly correlated systems: The massless gaussian field. *Journal of Statistical Physics*, 48:1249–1268, 01 1987.

[49] Nicolas Broutin, Thomas Duquesne, and Minmin Wang. Limits of multiplicative inhomogeneous random graphs and lévy trees: Limit theorems. `https://arxiv.org/abs/2002.02769`, 2020.

[50] Nicolas Broutin, Thomas Duquesne, and Minmin Wang. Limits of multiplicative inhomogeneous random graphs and lévy trees: The continuum graphs, 2020. `https://arxiv.org/abs/1804.05871`.

[51] Pietro Caputo and Matteo Quattropani. Mixing time of PageRank surfers on sparse random digraphs. `https://arxiv.org/abs/1905.04993`, 2019.

[52] Sourav Chatterjee. Stein's method for concentration inequalities. *Probab. Theory Related Fields*, 138(1-2):305–321, 2007.

[53] Jeff Cheeger. A lower bound for the smallest eigenvalue of the laplacian. In *Proceedings of the Princeton conference in honor of Professor S. Bochner*, pages 195–199, 1969.

[54] Kai Lai Chung. *Markov chains with stationary transition probabilities.* Die Grundlehren der mathematischen Wissenschaften, Bd. 104. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1960.

[55] Guillaume Conchon-Kerjan. Cutoff for random lifts of weighted graphs. `https://arxiv.org/abs/1908.02898`, 2019.

[56] Guillaume Conchon-Kerjan. Anatomy of a gaussian giant: supercritical level-sets of the free field on random regular graphs. `https://arxiv.org/abs/2102.10975`, 2021.

[57] Guillaume Conchon-Kerjan. Supercritical GFF percolation on a regular tree : anatomy of the infinite cluster. *To appear.*, 2021+.

[58] Guillaume Conchon-Kerjan and Christina Goldschmidt. The stable graph: the metric space scaling limit of a critical random graph with i.i.d. power-law degrees. `https://arxiv.org/abs/2002.04954`, 2020.

[59] Nicolas Curien. Random graphs - the local convergence point of view. `https://www.imo.universite-paris-saclay.fr/~curien/cours/cours-RG.pdf`, 2018.

[60] D. J. Daley and D. Vere-Jones. *An introduction to the theory of point processes. Vol. II.* Probability and its Applications (New York). Springer, New York, second edition, 2008. General theory and structure.

[61] Souvik Dhara, Remco van der Hofstad, Johan van Leeuwaarden, and Sanchayan Sen. Critical window for the configuration model: finite third moment degrees. *Electron. J. Probab.*, pages paper no. 16, 33 pp., 2017.

[62] Souvik Dhara, Remco van der Hofstad, Johan S. H. van Leeuwaarden, and Sanchayan Sen. Heavy-tailed configuration models at criticality. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 56(3):1515 – 1558, 2020.

[63] Persi Diaconis and Mehrdad Shahshahani. Generating a random permutation with random transpositions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 57(2):159–179, Jun 1981.

[64] Jian Ding, Eyal Lubetzky, and Yuval Peres. Anatomy of the giant component: the strictly supercritical regime. *European J. Combin.*, 35:155–168, 2014.

[65] Alexander Drewitz, Alexis Prévost, and Pierre-Françcois Rodriguez. The Sign Clusters of the Massless Gaussian Free Field Percolate on $\{Z^d$, d $\geq$slant 3$\}$ (and more). *Communications in Mathematical Physics*, page 1398, August 2018.

[66] Yotam Drier and Nathan Linial. Minors in lifts of graphs. *Random Structures Algorithms*, 29(2):208–225, 2006.

[67] Hugo Duminil-Copin, Subhajit Goswami, Aran Raoufi, Franco Severo, and Ariel Yadin. Existence of phase transition for percolation using the gaussian free field. *Duke Mathematical Journal*, 169(18):3539–3563, Dec 2020.

[68] Hugo Duminil-Copin, Subhajit Goswami, Pierre-François Rodriguez, and Franco Severo. Equality of critical parameters for percolation of gaussian free field level-sets. `https://arxiv.org/abs/2002.07735`, 2020.

[69] Thomas Duquesne. A limit theorem for the contour process of conditioned Galton-Watson trees. *Ann. Probab.*, 31(2):996–1027, 2003.

[70] Thomas Duquesne and Jean-François Le Gall. Random trees, Lévy processes and spatial branching processes. *Astérisque*, 281:vi+147, 2002.

[71] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[72] Rick Durrett. Probability: Theory and examples (5th edition). `https://services.math.duke.edu/~rtd/PTE/PTE5_011119.pdf`, 2019.

[73] Nathalie Eisenbaum, Haya Kaspi, Michael B. Marcus, Jay Rosen, and Zhan Shi. A ray-knight theorem for symmetric markov processes. *Ann. Probab.*, 28(4):1781–1796, 10 2000.

[74] P. Erdős and A. Rényi. On the evolution of random graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 5:17–61, 1960.

[75] P. Erdös and A. Rényi. On random graphs i. *Publicationes Mathematicae Debrecen*, 6:290, 1959.

[76] Lorenzo Federico. Critical scaling limits of the random intersection graph. `https://arxiv.org/abs/1910.13227`, 2019.

[77] David A. Freedman. On tail probabilities for martingales. *Ann. Probability*, 3:100–118, 1975.

[78] A. Gaudillière and C. Landim. A dirichlet principle for non reversible markov chains and some recurrence theorems. *Probability Theory and Related Fields*, 71(1):55–89, Feb 2001.

[79] E. N. Gilbert. Random graphs. *Ann. Math. Statist.*, 30:1141–1144, 1959.

[80] Lorenz A. Gilch. Rate of escape of random walks on regular languages and free products by amalgamation of finite groups. In *Fifth Colloquium on Mathematics and Computer Science, DMTCS Proc*, pages 2544–2560, 2008.

[81] Lorenz A. Gilch. Asymptotic entropy of random walks on regular languages over a finite alphabet. *Electron. J. Probab.*, 21:Paper No. 8, 42, 2016.

[82] Christina Goldschmidt, Bénédicte Haas, and Delphin Sénizergues. Stable graphs: distributions and line-breaking construction. `https://arxiv.org/abs/1811.06940`, 2018.

[83] Jonathan Hermon and Sam Olesker-Taylor. Cutoff for almost all random walks on abelian groups. `https://arxiv.org/abs/2102.02809`, 2021.

[84] Jonathan Hermon, Allan Sly, and Perla Sousi. Universality of cutoff for graphs with an added random matching. `https://arxiv.org/abs/2008.08564`, 2020.

[85] Jonathan Hermon and Sam Thomas. Cutoff for Mixing Times on Random Abelian Cayley Graphs. `https://arxiv.org/abs/1810.05130`, 2018.

[86] Markus Heydenreich and Remco van der Hofstad. *Progress in high-dimensional percolation and random graphs*. CRM Short Courses. Springer, Cham; Centre de Recherches Mathématiques, Montreal, QC, 2017.

[87] Shlomo Hoory, Nathan Linial, and Avi Wigderson. Expander graphs and their applications. *Bull. Amer. Math. Soc. (N.S.)*, 43(4):439–561, 2006.

[88] Svante Janson. Brownian excursion area, Wright's constants in graph enumeration, and other Brownian areas. *Probab. Surv.*, 4:80–145, 2007.

[89] Svante Janson. On percolation in random graphs with given vertex degrees. *Electron. J. Probab.*, 14:no. 5, 87–118, 2009.

[90] Svante Janson and Malwina J. Luczak. A new approach to the giant component problem. *Random Structures Algorithms*, 34(2):197–216, 2009.

[91] Adrien Joseph. The component sizes of a critical random graph with given degree sequence. *Ann. Appl. Probab.*, 24(6):2560–2594, 2014.

[92] Igor Kortchemski. Sub-exponential tail bounds for conditioned stable Bienaymé-Galton-Watson trees. *Probab. Theory Related Fields*, 168(1-2):1–40, 2017.

[93] Michael Krivelevich, Eyal Lubetzky, and Benny Sudakov. Asymptotics in percolation on high-girth expanders, 2020.

[94] Steven P. Lalley. Random walks on regular languages and algebraic systems of generating functions. In *Algebraic methods in statistics and probability (Notre Dame, IN, 2000)*, volume 287 of *Contemp. Math.*, pages 201–230. Amer. Math. Soc., Providence, RI, 2001.

[95] Jean-François Le Gall. Random trees and applications. *Probab. Surv.*, 2:245–311, 2005.

[96] Jean-François Le Gall. *Brownian motion, martingales, and stochastic calculus*, volume 274 of *Graduate Texts in Mathematics*. Springer, 2016.

[97] Yves Le Jan. Markov paths, loops and fields. *Lecture Notes in Mathematics -Springer-verlag-*, 2026, 08 2008.

[98] David A. Levin, Yuval Peres, Elizabeth L. Wilmer, and For The Reader. For the instructor.

[99] Nathan Linial and Eyal Rozenman. Random lifts of graphs: perfect matchings. *Combinatorica*, 25(4):407–424, 2005.

[100] Eyal Lubetzky and Yuval Peres. Cutoff on all ramanujan graphs. *Geometric and Functional Analysis*, 26, 07 2016.

[101] Eyal Lubetzky and Allan Sly. Cutoff phenomena for random walks on random regular graphs. *Duke Math. J.*, 153(3):475–510, 2010.

[102] Eyal Lubetzky, Benny Sudakov, and Van Vu. Spectra of lifted ramanujan graphs. *Advances in Mathematics*, 227(4):1612–1645, 2011.

[103] Alexander Lubotzky. *Discrete groups, expanding graphs and invariant measures*, volume 125 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1994. With an appendix by Jonathan D. Rogawski.

[104] Titus Lupu. From loop clusters and random interlacements to the free field. *Ann. Probab.*, 44(3):2117–2146, 05 2016.

[105] Russell Lyons. Random walks and percolation on trees. *Ann. Probab.*, 18(3):931–958, 1990.

[106] Terry Lyons. A simple criterion for transience of a reversible Markov chain. *Ann. Probab.*, 11(2):393–402, 1983.

[107] Jean-François Marckert and Abdelkader Mokkadem. The depth first processes of Galton-Watson trees converge to the same Brownian excursion. *Ann. Probab.*, 31(3):1655–1678, 2003.

[108] James B. Martin and Balázs Ráth. Rigid representations of the multiplicative coalescent with linear deletion. *Electron. J. Probab.*, 22:Paper No. 83, 47, 2017.

[109] Grégory Miermont. Self-similar fragmentations derived from the stable tree. I. Splitting at heights. *Probab. Theory Related Fields*, 127(3):423–454, 2003.

[110] Grégory Miermont. Self-similar fragmentations derived from the stable tree. II. Splitting at nodes. *Probab. Theory Related Fields*, 131(3):341–375, 2005.

[111] S. A. Molchanov and A. K. Stepanov. Percolation in random fields. i. *Theoretical and Mathematical Physics*, 55(2):478–484, 1983.

[112] Michael Molloy and Bruce Reed. A critical point for random graphs with a given degree sequence. *Random Structures & Algorithms*, 6(2-3):161–180, 1995.

[113] Michael Molloy and Bruce Reed. The size of the giant component of a random graph with a given degree sequence. *Combin. Probab. Comput.*, 7(3):295–305, 1998.

[114] Ravi Montenegro and Prasad Tetali. Mathematical aspects of mixing times in Markov chains. *Found. Trends Theor. Comput. Sci.*, 1(3):x+121, 2006.

[115] Asaf Nachmias and Yuval Peres. Critical percolation on random regular graphs. *Random Structures Algorithms*, 36(2):111–148, 2010.

[116] Tatiana Nagnibeda, Stockholm Sweeden, and A M S-t. Green functions on trees with finitely many cone types, 1999.

[117] Tatiana Nagnibeda and Wolfgang Woess. Random walks on trees with finitely many cone types. *J. Theoret. Probab.*, 15(2):383–422, 2002.

[118] Mark Newman. Newman mej.. the structure and function of complex networks. siam rev 45: 167-256. *SIAM Review*, 45, 08 2003.

[119] Ilkka Norros and Hannu Reittu. On a conditionally Poissonian graph process. *Adv. in Appl. Probab.*, 38(1):59–75, 2006.

[120] Jim Pitman and Ngoc M. Tran. Size-biased permutation of a finite sequence with independent and identically distributed terms. *Bernoulli*, 21(4):2484–2512, 11 2015.

[121] Serguei Popov and Balazs Rath. On decoupling inequalities and percolation of excursion sets of the gaussian free field. *Journal of Statistical Physics*, 159, 07 2013.

[122] Oliver Riordan. The phase transition in the configuration model. *Combin. Probab. Comput.*, 21(1-2):265–299, 2012.

[123] Pierre-François Rodriguez and Alain-Sol Sznitman. Phase transition and level-set percolation for the gaussian free field. *Communications in Mathematical Physics*, 320(2):571–601, 2013.

[124] L.C.G. Rogers. A new identity for real levy processes. *Annales de l'I.H.P. Probabilités et statistiques*, 20(1):21–34, 1984.

[125] C. Sabot and P. Tarrès. Inverting ray-knight identity. *Probability Theory and Related Fields*, 165(3):559–580, Aug 2016.

[126] Laurent Saloff-Coste. *Lectures on finite Markov chains*, pages 301–413. Springer Berlin Heidelberg, Berlin, Heidelberg, 1997.

[127] Stanley Sawyer and Tim Steger. The rate of escape for anisotropic random walks in a tree. *Probability Theory and Related Fields*, 76(2):207–230, Oct 1987.

[128] Perla Sousi and Sam Thomas. Cutoff for random walk on dynamical Erdős–Rényi graph. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 56(4):2745 – 2773, 2020.

[129] Johan S. H. van Leeuwaarden Souvik Dhara, Remco van der Hofstad. Critical percolation on scale-free random graphs: New universality class for the configuration model. *Communications in Mathematical Physics*, 382(1):123–171, 2021.

[130] Alain-Sol Sznitman. Random interlacements and the gaussian free field. *The Annals of Probability*, 40, 02 2011.

[131] Alain-Sol Sznitman. An isomorphism theorem for random interlacements. *Electron. Commun. Probab.*, 17:9 pp., 2012.

[132] Alain-Sol Sznitman. Coupling and an application to level-set percolation of the gaussian free field. *Electron. J. Probab.*, 21:26 pp., 2016.

[133] Christiane Takacs. Random walk on periodic trees. *Electron. J. Probab.*, 2:16 pp., 1997.

[134] Tatyana S. Turova. Diffusion approximation for the components in critical inhomogeneous random graphs of rank 1. *Random Structures Algorithms*, 43(4):486–539, 2013.

[135] Remco van der Hofstad. *Random graphs and complex networks: volume I.* Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2016.

[136] Jiří Černý. Level-set percolation of the Gaussian free field on regular graphs III: giant component on expanders. *arXiv e-prints*, page arXiv:2105.13974, May 2021.

[137] Wolfgang Woess. Random walks and periodic continued fractions. *Adv. in Appl. Probab.*, 17(1):67–84, 1985.

[138] Wolfgang Woess. *Random Walks on Infinite Graphs and Groups*, volume 138 of *Cambridge Tracts in Mathematics.* Cambridge University Press, United Kingdom, paperback re-edition (with corrections) edition, 2008. 352 S.

[139] Nicholas Wormald. *Some problems in the enumeration of labelled graphs*. PhD thesis, University of Newcastle, Australia, 1978.

[140] Jiří Černý, Augusto Teixeira, and David Windisch. Giant vacant component left by a random walk in a random d -regular graph. *Ann. Inst. H. Poincaré Probab. Statist.*, 47(4):929–968, 11 2011.